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Distality Rank

Roland Walker

University of Illinois at Chicago

2020 North American Annual Meeting of the ASL
Special Session on Model Theory

Understanding Unstable NIP Theories

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Understanding Unstable NIP Theories

- Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.
- It was motivated as an attempt to better understand unstable NIP theories by studying their stable and "purely unstable," or ***distal***, parts separately. This decomposition is particularly easy to see for algebraically closed valued fields:

Stable Part:	Residue field
Distal Part:	Value group

Distal NIP Theories

Distality quickly became interesting and useful in its own right, and much progress has been made in recent years studying distal NIP theories. Such a theory exhibits **no stable behavior** since it is dominated by its order-like component.

Examples:

- o-minimal theories
- p-adics
- certain expansions of o-minimal theories (Hieronymi, Nell 2017)
- the asymptotic couple of the field of logarithmic transseries (Gehret, Kaplan 2018)

Combinatorial Results

Many classical combinatorial results can be improved when study is restricted to objects definable in distal NIP structures.

- Cutting Lemma (Chernikov, Galvan, Starchenko 2018)

  > “ We believe that distal structures provide the most general natural setting for investigating questions in ‘generalized incidence combinatorics.’ ”

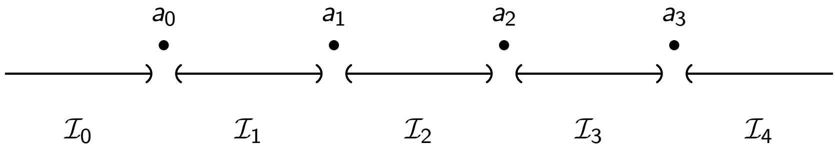
- (p, q)-Theorem (Boxall, Kestner 2018)

- Szemerédi Regularity Lemma (Chernikov, Starchenko 2018)
  - Polynomial bound on partition size
  - Homogeneity

m-Distality

1-Distality in Pictures...

A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is ***1-distal*** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each **singleton** from A inserts indiscernibly...



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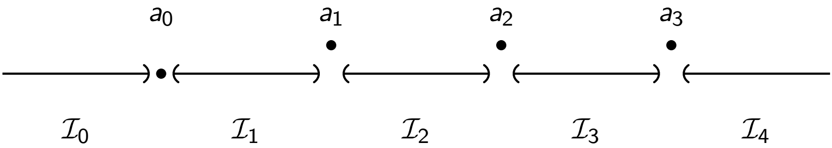

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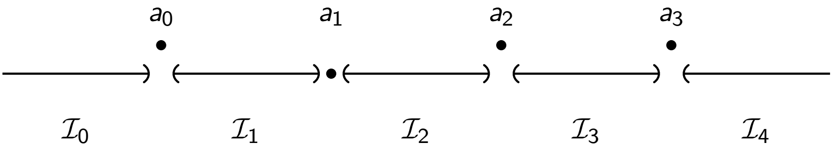

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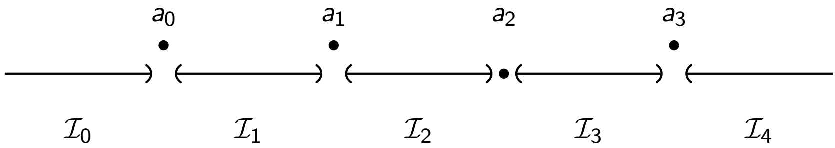

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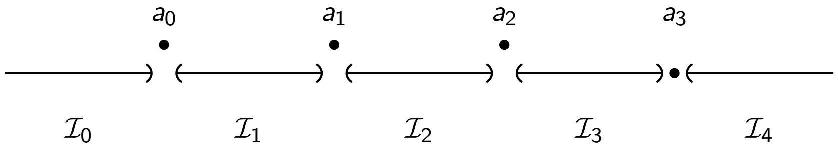

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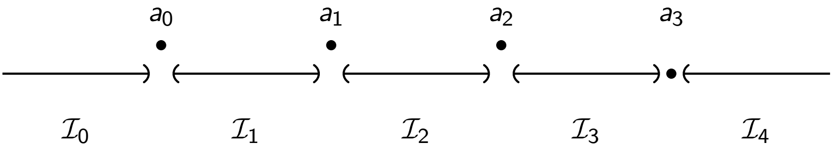

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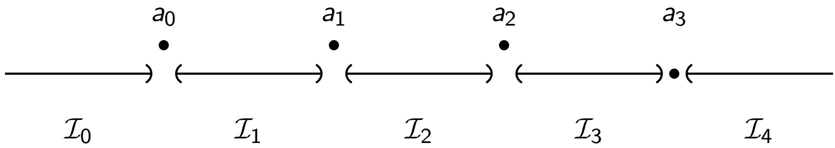

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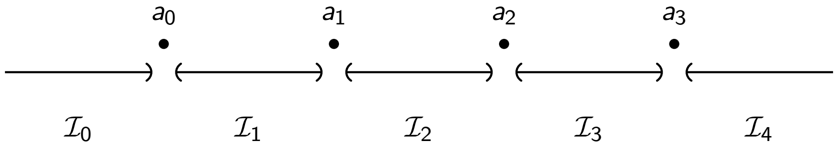

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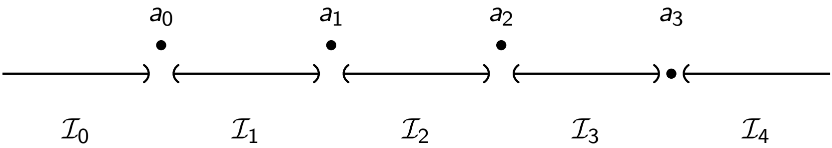

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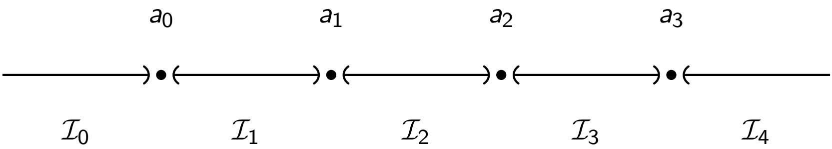

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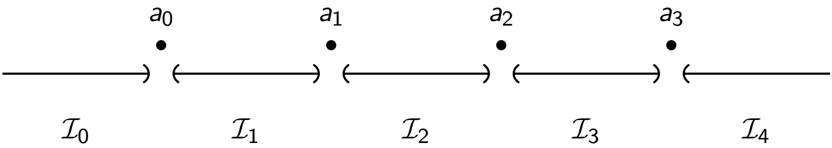

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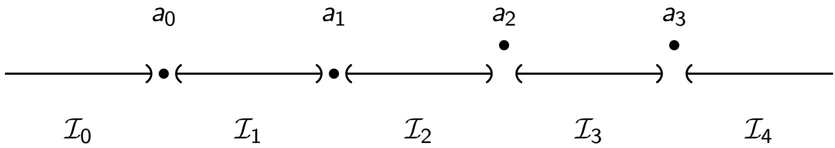

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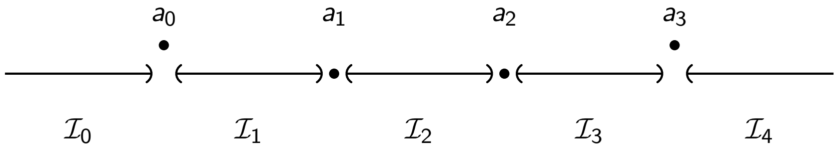

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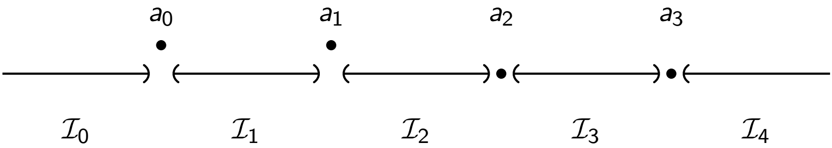

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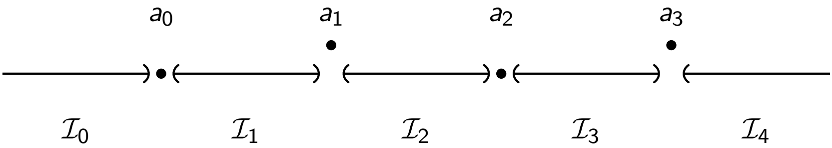

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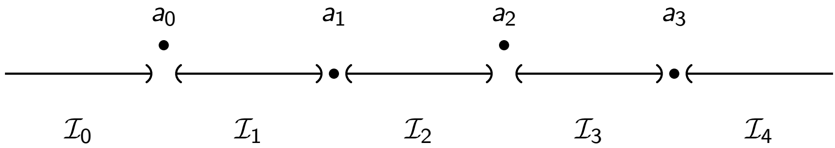

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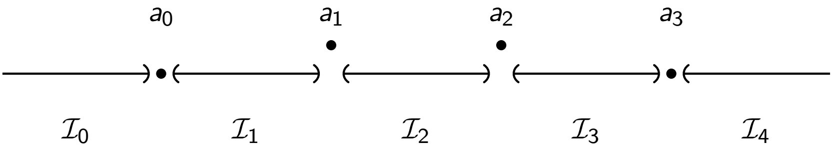

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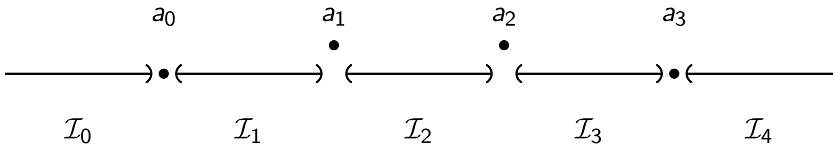

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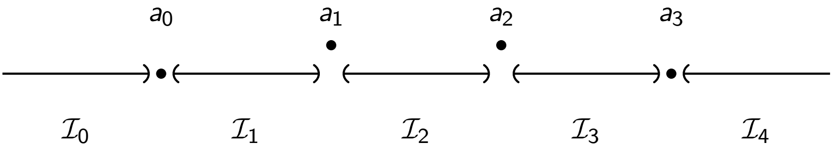

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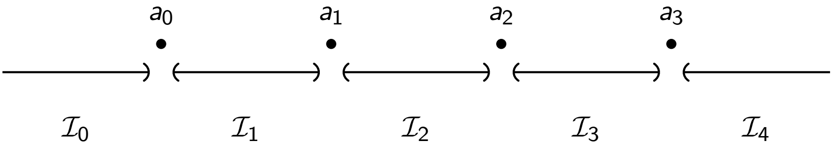

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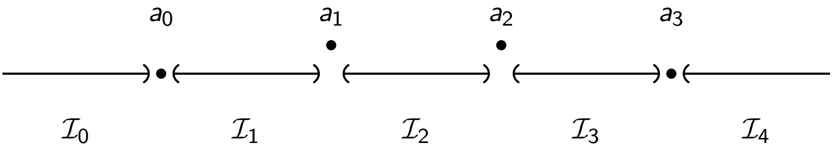

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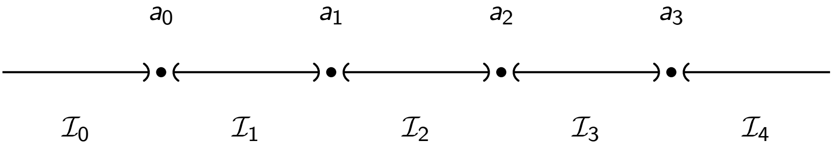

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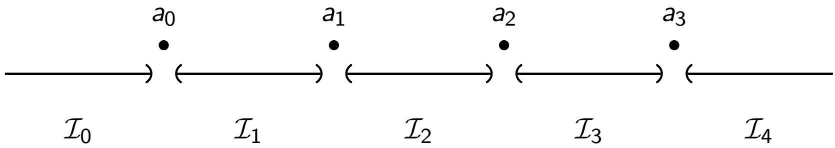

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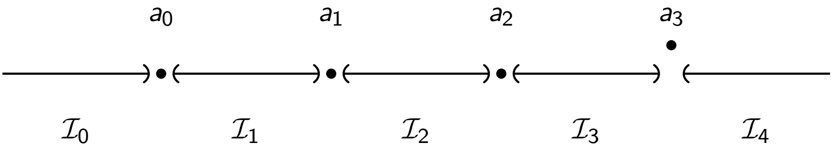

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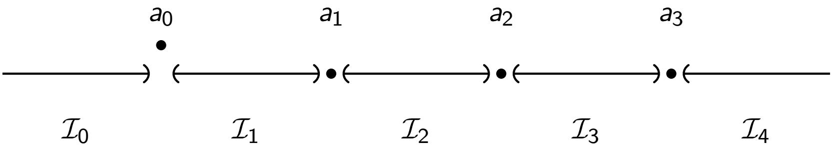

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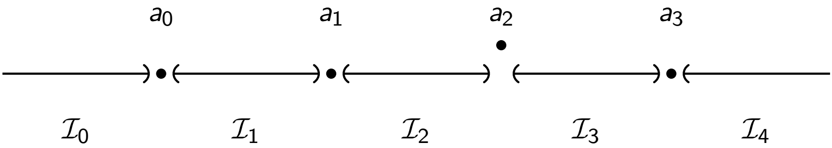

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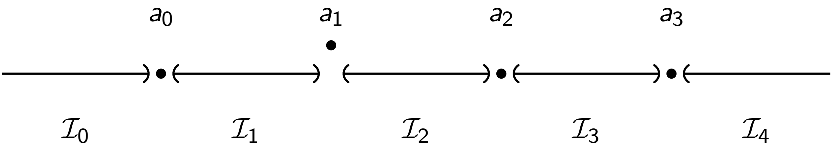

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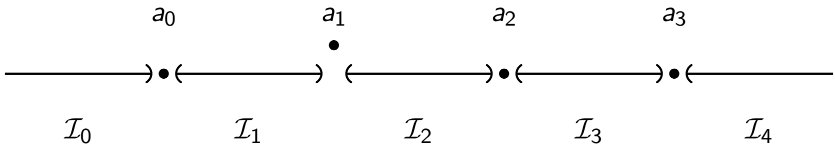

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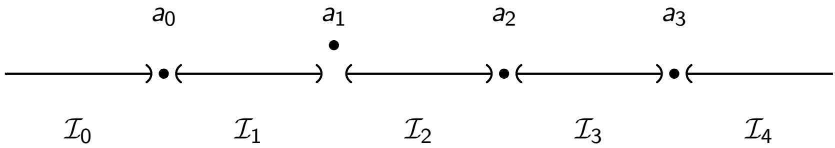

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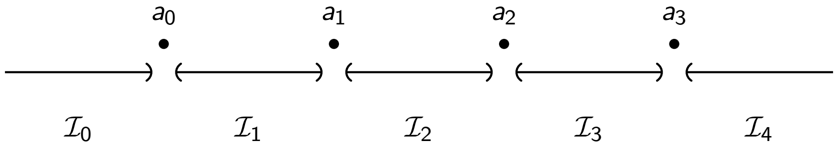

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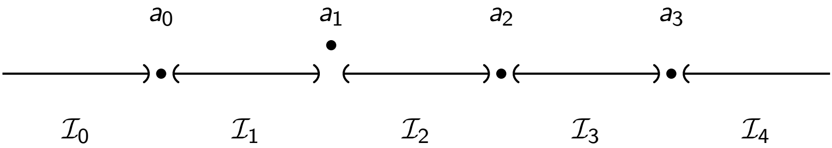

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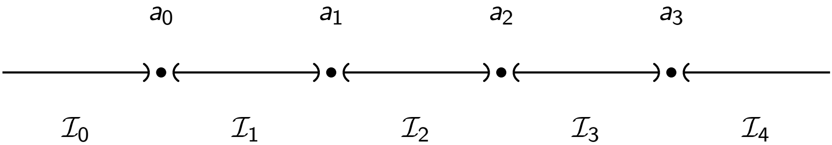

m-Distality

Let $n > m > 0$.

Definition

We say a Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_n$ is ***m-distal*** iff: for all sets $A = (a_0, \ldots, a_{n-1}) \subseteq U$, if A does not insert indiscernibly into $\mathcal{I}$, then some m-element subset of A does not insert indiscernibly into $\mathcal{I}$.

m-Distality for EM-types

Let $n > m > 0$.

Definition

A complete EM-type Γ is (n, m)-***distal*** iff: every Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_n \models^{\mathrm{EM}} \Gamma$ is m-distal.

Lemma

If Γ is $(m+1, m)$-distal, then Γ is (n, m)-distal for all $n > m$.

Proof: Induction on n. ■

m-Distality for EM-types

Definition

A complete EM-type Γ is ***m*-distal** iff: it is $(m+1, m)$-distal.

Theorem

Suppose T is NIP. A complete EM-type Γ is m-distal if and only if there is an m-distal Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\mathrm{EM}} \Gamma$.

Distality Rank for EM-Types

Observation: If a complete EM-type Γ is m-distal, then it is also n-distal for all $n > m$.

Definition

The ***distality rank*** of a complete EM-type Γ, written $\mathrm{DR}(\Gamma)$, is the least $m \geq 1$ such that Γ is m-distal. If no such finite m exists, we say the distality rank of Γ is ω.

Distality Rank for Theories

Let $m > 0$.

Definition

A theory T, not necessarily complete, is ***m-distal*** iff: for all completions of T and all tuple sizes κ, every $\Gamma \in S^{\mathrm{EM}}(\kappa \cdot \omega)$ is m-distal.

In the existing literature, a theory is called distal if and only if it is 1-distal.

Definition

The ***distality rank*** of a theory T, written $\mathrm{DR}(T)$, is the least $m \geq 1$ such that T is m-distal. If no such finite m exists, we say the distality rank of T is ω.

Finding Examples...

Proposition

Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If T is an $\mathcal{L}$-theory with quantifier elimination, then $\mathrm{DR}(T) \leq m$.

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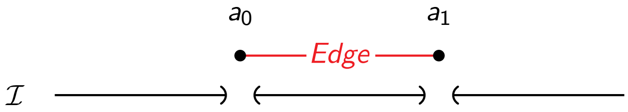
Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If T is an $\mathcal{L}$-theory with quantifier elimination, then $\mathrm{DR}(T) \leq m$.

This corollary helps us find examples by putting an upper bound on distality rank:

- The theory of the random graph has distality rank 2.



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- The theory of the random graph has distality rank 2.
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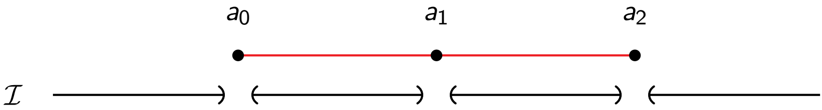

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  This generalizes, so...
- The theory of the random m-(hyper)graph has distality rank m.

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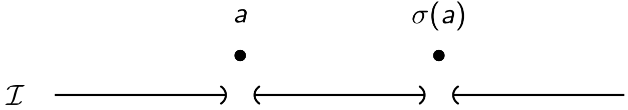
This corollary helps us find examples by putting an upper bound on distality rank:

- The theory of the random m-(hyper)graph has distality rank m.
- The theories of $(\mathbb{N}, \sigma, 0)$ and $(\mathbb{Z}, \sigma)$, where $\sigma : x \mapsto x + 1$, have distality rank 2.



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We can not apply the proposition to groups...

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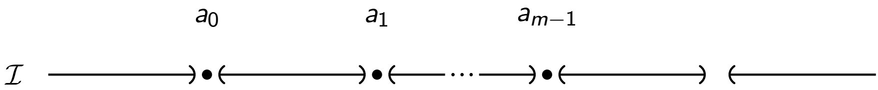

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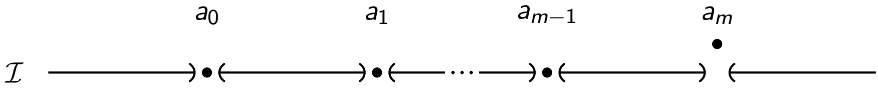

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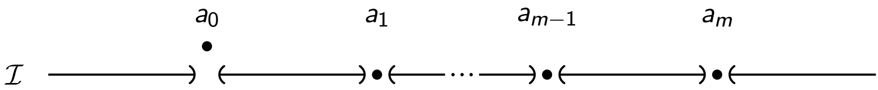

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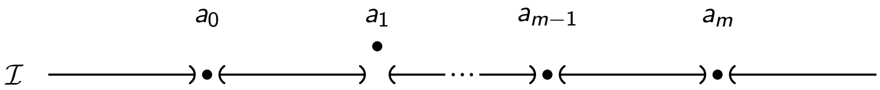

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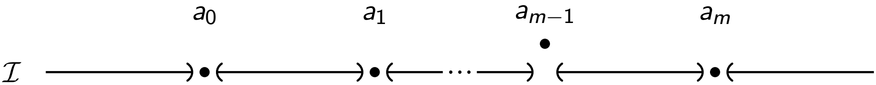

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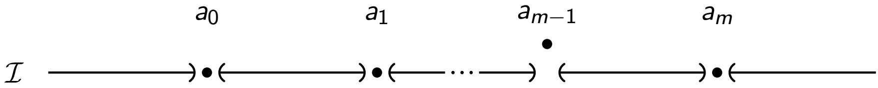

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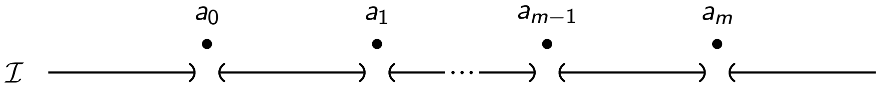

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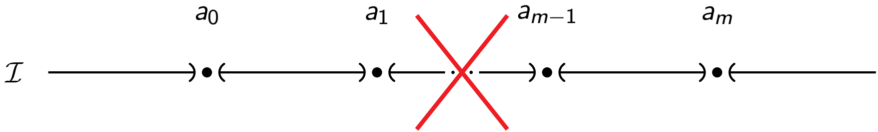

Base Change

Adding named parameters does not increase distality rank...

Proposition

If T is a complete theory and $B \subseteq U$ is a small set of parameters, then $\mathrm{DR}(T_B) \leq \mathrm{DR}(T)$.

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If T is NIP, adding named parameters does not change distality rank...

Base Change Theorem

If T is NIP and $B \subseteq U$ is a small set of parameters, then $\mathrm{DR}(T_B) = \mathrm{DR}(T)$.

Type Determinacy

Let $n > m > 0$.

Definition

Given $p \in S_A(x_0, \ldots, x_{n-1})$, we say that the n-type p is ***m-determined*** iff: it is completely determined by the m-types

$$\{q \in S_A(x_{i_0}, \ldots, x_{i_{m-1}}) : q \subseteq p \text{ and } i_0 < \cdots < i_{m-1} < n\}$$

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Theorem

If T is m-distal, then for any n global invariant types

$$p_0(x_0), \ldots, p_{n-1}(x_{n-1})$$

which commute pairwise, their product $p_0 \otimes \cdots \otimes p_{n-1}$ is m-determined.

Furthermore*, if T is NIP, the converse holds as well.*

Relationship between m-Distality and m-Dependence

Shelah introduced ***m*-dependence** as a property of first-order theories (and formulae) which generalizes NIP:

- 1-dependence $\iff$ NIP
- m-dependence $\implies$ $(m+1)$-dependence

New result courtesy of Artem Chernikov:

- m-distality $\implies$ m-dependence

Conjecture:

- m-distal regularity improves m-dependent regularity

Thank You!

A link to the paper and a longer version of the slides can be found at my website...

https://homepages.math.uic.edu/~roland/