Effective Notions of Weak Convergence of Measures on the Real Line

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Outline

I. Background

- Weak Convergence of Measures
- Computable Analysis
- Computable Measure Theory
- Questions
- Goals

II. Effective Weak Convergence of Measures

- Definitions
- Examples
- Results

Part I: Background

Weak Convergence of Measures

- A sequence of finite Borel measures $\{\mu_n\}_{n\in\mathbb{N}}$ on a metric space X *weakly converges* to a measure μ if the integrals $\{\int_X f\,d\mu_n\}_{n\in\mathbb{N}}$ converge to $\int_X f\,d\mu$ for any bounded continuous $f : X \to \mathbb{R}$.
- Prokhorov metric on finite Borel measures: $\rho(\mu,\nu) :=$ the infimum over all $\epsilon > 0$ so that $\mu(A) \leq \nu(A^\epsilon) + \epsilon$ and $\nu(A) \leq \mu(A^\epsilon) + \epsilon$ for all $A \in \mathcal{B}(X)$, where
  - $\mathcal{B}(X)$ is the Borel σ-algebra of X;
  - $A^\epsilon = \bigcup_{a\in A} B(a,\epsilon)$;
  - $B(a,\epsilon)$ is the open ball of radius ϵ around a.
- If X is a separable metric space, the space $\mathcal{M}(X)$ of finite Borel measures on X becomes a separable metric space under the Prokhorov metric.

Computable Analysis

- A *computable metric space* is a triple (X, d, S) with the following properties:
  - (X, d) is a complete separable metric space
  - $S = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X
  - $d(s_i, s_j)$ is computable uniformly in i, j
- Examples:
  - $(\mathbb{R}, |\cdot|, \mathbb{Q})$
  - $(2^\omega, d_C, S_C)$ where $d_C(X, Y) = 2^{-\min\{n : X(n) \neq Y(n)\}}$ and $S_C = \{\sigma 0^\omega : \sigma \in 2^{<\omega}\}$
- Throughout the talk, we will focus on $X = \mathbb{R}$.

- A *Cauchy name* of $x \in \mathbb{R}$ is a computable sequence of rationals $\{q_n\}_{n\in\mathbb{N}}$ so that $|q_n - q_{n+1}| < 2^{-n}$.

- A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable* if there is a Turing functional that sends a Cauchy name of $x \in \operatorname{dom} f$ to a Cauchy name of $f(x)$.

- A *compact-open (c.o.)* name of a function $f \in C(\mathbb{R})$ is an enumeration $\rho_f \in \Sigma^\omega$ of the set $\{N_{I,J} : f \in N_{I,J}\}$, where
  - Σ is a finite alphabet;
  - $I \subseteq \mathbb{R}$ is a compact interval;
  - $J \subseteq \mathbb{R}$ is an open interval;
  - $N_{I,J} = \{f \in C(\mathbb{R}) : f(I) \subseteq J\}$.

- A function $F :\subseteq C(\mathbb{R}) \to \mathbb{R}$ is *computable* if there is a Turing functional that sends a c.o. name of $f \in \operatorname{dom} F$ to a Cauchy name of $F(f)$.

Computable Measure Theory

- A measure $\mu \in \mathcal{M}(\mathbb{R})$ is *computable* if $\mu(\mathbb{R})$ is computable and $\mu(U)$ is left-c.e. uniformly from (an index of) a Σ^0_1 subset U of $\mathbb{R}$.
- A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R})$ is *uniformly computable* if μ_n is a computable measure uniformly in n.

Computable Measure Theory

Work by M. Hoyrup and C. Rojas[1] gives us the following:

- $(\mathcal{M}(\mathbb{R}), \rho, \mathcal{D})$ is a computable metric space, where $\mathcal{D}$ denotes the space of finite rational linear combinations of Dirac measures on $\mathbb{R}$.
- $\mu \in \mathcal{M}(\mathbb{R})$ is computable if and only if $I_\mu : f \mapsto \int_{\mathbb{R}} f d\mu$ is computable on computable $f \in C(\mathbb{R})$, uniformly from (a c.o. name of) f.

[1] M. Hoyrup and C. Rojas. "Computability of probability measures and Martin-Löf randomness over metric spaces". In: *Information and Computation* 207 (2009), pp. 830–847.

Question.

Given a computable metric space X and a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(X)$, what are the necessary and sufficient conditions for $\{\mu_n\}_{n\in\mathbb{N}}$ to *effectively* weakly converge to a computable measure $\mu \in \mathcal{M}(X)$?

Before addressing this question, we must first answer the following:

Question.

Given a computable metric space X, what does it mean for a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(X)$ to *effectively* weakly converge?

Goals

Goals for this talk:

- Define *effective weak (e.w.) convergence* and *uniformly effective weak (u.e.w.) convergence* in $\mathcal{M}(\mathbb{R})$
- Show that sequences in $\mathcal{M}(\mathbb{R})$ that are uniformly computable and e.w. convergent tend to a computable measure in $\mathcal{M}(\mathbb{R})$
- Distinguish e.w. convergence from classical weak convergence
- Show that e.w. and u.e.w. convergence in $\mathcal{M}(\mathbb{R})$ are equivalent for uniformly computable sequences

Part II: Effective Weak Convergence of Measures

Definitions

- Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathbb{R})$ and Σ be a finite alphabet.

Definition.

We say that $\{\mu_n\}_{n\in\mathbb{N}}$ *effectively weakly (e.w.) converges* to a finite Borel measure μ if there exists a (partial) computable function $g :\subseteq \mathbb{N}^2 \to \mathbb{N}$ such that for any $e, N, n \in \mathbb{N}$, if the eth Turing functional computes a c.o. name $\rho_f \in \Sigma^\omega$ of a function $f \in C_b(\mathbb{R})$ with $|f| < e$, then $g(e,N)\downarrow$, and

$$n \geq g(e,N) \Rightarrow \left| \int_{\mathbb{R}} f\, d\mu_n - \int_{\mathbb{R}} f\, d\mu \right| < 2^{-N}.$$

We call g an *effective modulus of convergence (e.m.o.c.)* for $\{\mu_n\}_{n\in\mathbb{N}}$.

Definition.

We say that $\{\mu_n\}_{n\in\mathbb{N}}$ *uniformly effectively weakly (u.e.w.) converges* to a finite Borel measure μ if there exists a (partial) computable function $g :\subseteq \Sigma^\omega \times \mathbb{N}^2 \to \mathbb{N}$ such that for any c.o. name $\rho_f \in \Sigma^\omega$ of a function $f \in C_b(\mathbb{R})$ and any $B, N, n \in \mathbb{N}$, $|f| \leq B$ implies $g(\rho_f, B, N)\downarrow$, and

$$n \geq g(\rho_f, B, N) \Rightarrow \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| < 2^{-N}.$$

We call g a *uniformly effective modulus of convergence (u.e.m.o.c)* for $\{\mu_n\}_{n\in\mathbb{N}}$.

Examples

- Fix $a, b \in \mathbb{Q}$, $E \in \mathcal{B}(\mathbb{R})$, a uniformly computable sequence $\{q_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ that decreases to 0. The sequence $\mu_n(E) = \lambda(E \cap [a - q_n, b + q_n])$ e.w. converges to $\mu(E) = \lambda(E \cap [a, b])$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

- For a uniformly computable sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ that converges to some computable $r \in \mathbb{R}$, the sequence of Dirac measures $\{\delta_{r_n}\}_{n\in\mathbb{N}}$ e.w. converges to δ_r.

Results

Proposition. (R. 2020+)

If $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable and e.w. converges to μ, then μ is a computable measure.

Proof.

It suffices to show that $f \mapsto \int_{\mathbb{R}} f\,d\mu$ is a computable operator on computable $f : \mathbb{R} \to [0,1]$. Let g be an e.m.o.c. for $\{\mu_n\}_{n\in\mathbb{N}}$. For computable $f : \mathbb{R} \to [0,1]$ and $t \in \mathbb{N}$, we may find an index $e_f > 1$ of a Turing functional that computes a c.o. name of f with $|f| < e_f$, so $g(e_f, t+1)\downarrow = n_0$. Then, compute a $q \in \mathbb{Q}$ that approximates $\int_{\mathbb{R}} f\,d\mu_{n_0}$ to within 2^{-t-1} since $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. The result follows by definition of e.m.o.c. and the triangle inequality. □

Proposition. (R. 2020+)

There exists a sequence of computable measures $\{\mu_n\}_{n\in\mathbb{N}}$ that weakly converges but does not e.w. converge.

Proof.

Let $\{q_n\}_{n\in\mathbb{N}}$ be a uniformly computable increasing sequence in $\mathbb{Q}$ that converges to an incomputable left-c.e. $\alpha \in \mathbb{R}$. For $E \in \mathcal{B}(\mathbb{R})$ and λ Lebesgue measure on $\mathcal{B}(\mathbb{R})$, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ defined by $\mu_n(E) = \lambda(E \cap [0, q_n])$ weakly converges to $\mu(E) = \lambda(E \cap [0, \alpha])$, but fails to e.w. converge since $\mu(\mathbb{R}) = \lambda([0, \alpha]) = \alpha$ is not computable. □

Results

Theorem. (R. 2020+)

Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. The following are equivalent:

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ is e.w. convergent;

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ is u.e.w. convergent.

Proof Sketch.
(2) $\Rightarrow$ (1): For $\overline{g}$ a u.e.m.o.c. of $\{\mu_n\}_{n\in\mathbb{N}}$, let $g(e, N)\downarrow = \overline{g}(P_e, e, N)$, where $\{P_e\}_{e\in\mathbb{N}}$ is an effective enumeration of all computable sequences in $\Sigma^{\leq\omega}$. □

Proof Sketch.
(1) $\Rightarrow$ (2): Fix $f \in C_b(\mathbb{R})$ with c.o. name $\rho_f \in \Sigma^{\leq\omega}$, $B \in \mathbb{N}$ with $|f| \leq B$, and $N \in \mathbb{N}$. For $\overline{g}$ an e.m.o.c. of $\{\mu_n\}_{n\in\mathbb{N}}$, define the following effective procedure for $g :\subseteq \Sigma^\omega \times \mathbb{N}^2 \to \mathbb{N}$:

1. Compute $a, n_0 \in \mathbb{N}$ so that $\mu([-a,a]^c) < 2^{-N-4}/B$ and $\mu_n([-a,a]^c) < 2^{-N-4}/B$ for every $n \geq n_0$.
2. Compute $\psi \in P_\mathbb{Q}[-a,a]$ and $n' \in \mathbb{N}$ with $n' \geq n_0$ so that $\left|\int_\mathbb{R}(f-\psi)d\mu\right| < 2^{-N-2}$ and $\left|\int_\mathbb{R}(f-\psi)d\mu_n\right| < 2^{-N-2}$ for every $n \geq n'$.
3. Search for an index $e \in \mathbb{N}$ of a Turing machine that computes a c.o. name for ψ with $|\psi| < e$, and let $g(\rho_f, B, N)\downarrow = \overline{g}(e, N+1)$.

□

Thank you!

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