How to obtain lower bounds in set theory

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Mar 26, 2020

2020 North American Meeting of the ASL

Inner Model Theory

Determinacy

Combinatorics of κ-Trees

Looking beyond

Inner Model Theory

The main goal of inner model theory is to construct L-like models, which we call mice, for stronger and stronger large cardinals.

Gödel's constructible universe L

Definition

Let E be a set or a proper class. Let

$$
\begin{aligned}
J_0[E] &= \emptyset \\
J_{\alpha+\omega}[E] &= \mathrm{rud}_E(J_\alpha[E] \cup \{J_\alpha[E]\}) \\
J_{\omega\lambda}[E] &= \bigcup_{\alpha<\lambda} J_{\omega\alpha}[E] \text{ for limit } \lambda \\
L[E] &= \bigcup_{\alpha\in\mathrm{Ord}} J_{\omega\alpha}[E]
\end{aligned}
$$

Note that rud_E denotes the closure under functions which are rudimentary in E (i.e. basic set operations like minus, union and pairing or intersection with E).

Basic properties of L

Condensation Let α be an infinite limit ordinal and let

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Then the transitive collapse of M is equal to J_β for some limit ordinal $\beta \leq \alpha$.

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Comparison Let J_α and J_β for limit ordinals α and β be initial segments of L. Then one is an initial segment of the other, that means

$$J_\alpha \trianglelefteq J_\beta \text{ or } J_\beta \trianglelefteq J_\alpha.$$

A First Equivalence for Analytic Determinacy

Definition

Let x be a real. We say $x^\#$ exists iff for some limit ordinal λ, the model $J_{\omega\lambda}[x]$ has an uncountable set Γ of indiscernibles, i.e. for $n < \omega$ and any two increasing sequences $(\alpha_0, \ldots, \alpha_n)$ and $(\beta_0, \ldots, \beta_n)$ from Γ and any formula φ,

$$J_{\omega\lambda}[x] \models \varphi(x, \alpha_0, \ldots, \alpha_n) \Leftrightarrow J_{\omega\lambda}[x] \models \varphi(x, \beta_0, \ldots, \beta_n).$$

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The following are equivalent.

(a) *All analytic games are determined.*

(b) $x^\#$ *exists for all reals* x.

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The following are equivalent.

(a) *All analytic games are determined.*

(b) $x^\#$ *exists for all reals* x.

To see how this relates to measurable cardinals, we need to look at a different definition of $x^\#$.

Basic Concepts of Inner Model Theory

Definition

Let $\mathcal{M}$ be a transitive model of set theory, κ a cardinal in $\mathcal{M}$ and $\mathcal{U}$ a κ-complete, nonprincipal ultrafilter on $\mathcal{M}$. Then there is a transitive model $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, \mathcal{U})$ and an elementary embedding $i_{\mathcal{U}} : \mathcal{M} \to \mathcal{N}$ with critical point κ. We call $\mathcal{N}$ the *ultrapower* of $\mathcal{M}$ via $\mathcal{U}$.

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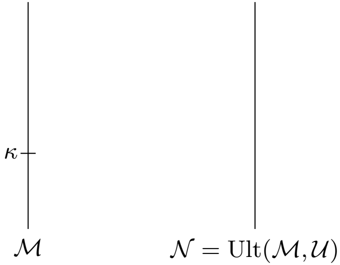

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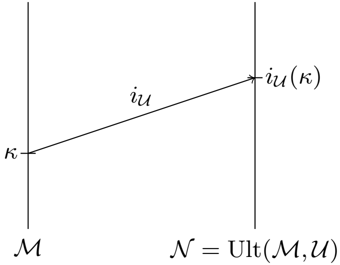

$x^{\#}$ as a mouse

Definition (Equivalent definition of $x^{\#}$)

If it exists, let $x^{\#}$ be the unique premouse of the form $\mathcal{M} = (J_\alpha[x], \in, U)$ with $\mathrm{crit}(U) = \kappa$ such that

1. if $z \subset \mathcal{P}(\kappa)$ is in $\mathcal{M}$ with $|z|^{\mathcal{M}} = \kappa$, then $U \cap z \in \mathcal{M}$,
2. $\mathcal{M} \vDash U$ is a non-trivial normal κ-complete ultrafilter on κ,
3. a fine structural condition which implies that $(J_{\alpha+\omega}[x], \in, U) \vDash |\alpha| = \omega$, and
4. every countable linear iterate of $\mathcal{M}$ (via U) is well-founded.

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Note: The images of the critical point κ of the external measure U (when iterating the model $x^\#$ linearly) form an uncountable set of indiscernibles Γ for some large enough $J_{\omega\lambda}[x]$.

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Mitchell and Jensen generalized the concept of measures to extenders to obtain stronger ultrapowers.

Definition

Let $\mathcal{M}$ be a countable model of set theory. An *extender* over $\mathcal{M}$ is a system of ultrafilters whose ultrapowers form a directed system, such that they give rise to a single elementary embedding.

In fact for every embedding $j : \mathcal{M} \to \mathcal{N}$ there is an extender E over $\mathcal{M}$ which gives rise to this embedding.

Comparison

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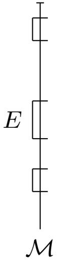

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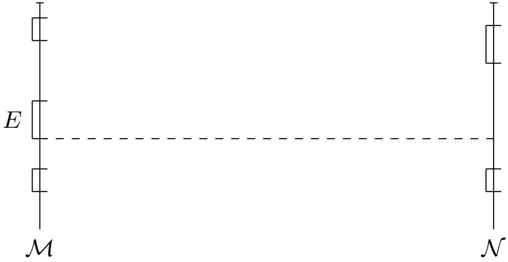

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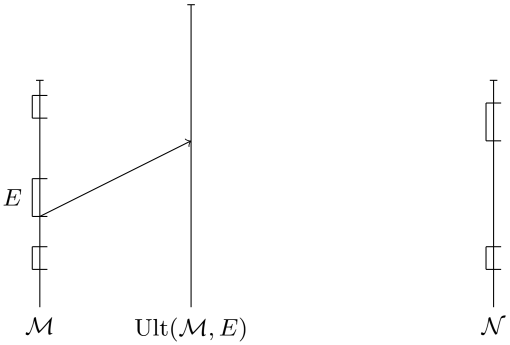

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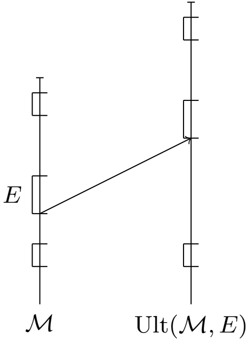

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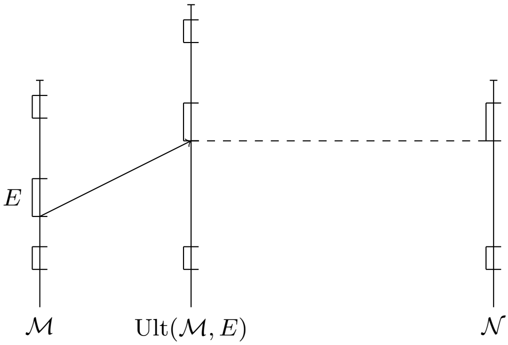

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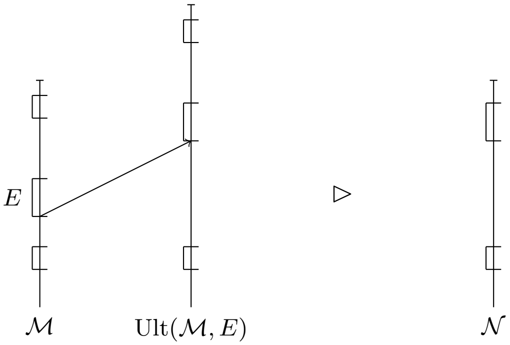

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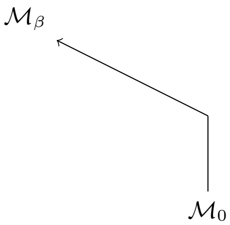

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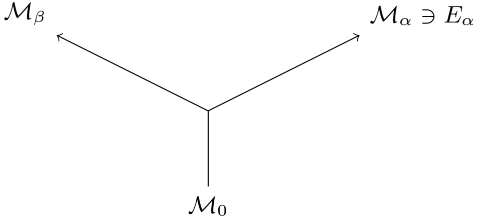

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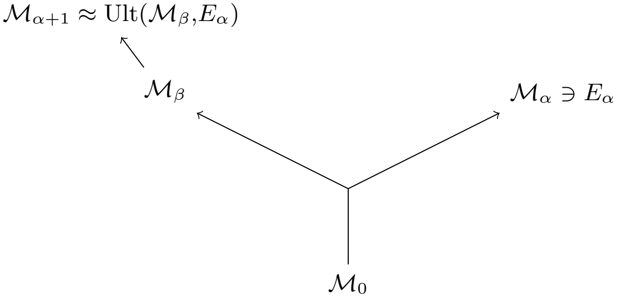

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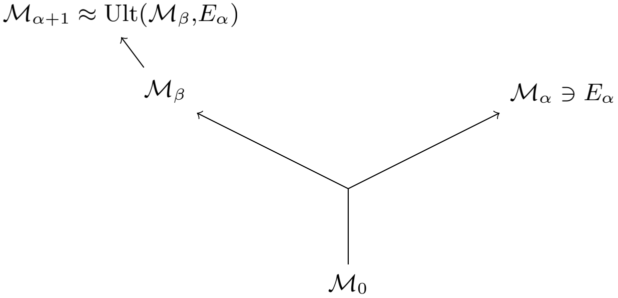

The central problem is to choose a cofinal branch such that the direct limit is well-founded.

The iteration game

More precisely we consider the following two player game $\mathcal{G}(\mathcal{M}, \omega_1)$ of length $< \omega_1$ for a premouse $\mathcal{M}$.

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Player II plays at the limit steps: Choose a cofinal well-founded branch of the iteration tree constructed so far.

Then player II wins if all the models constructed in the game are well-founded.

Definition

We say a premouse $\mathcal{M}$ is ω_1-iterable iff player II has a winning strategy in the game $\mathcal{G}(\mathcal{M}, \omega_1)$. This winning strategy is called an iteration strategy for $\mathcal{M}$.

Inner Model Theory

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Looking beyond

Games in Set Theory

Definition (Gale-Stewart 1953)

Let $A \subset \omega^\omega$. We denote the following game by $G(A)$

$$\begin{array}{c|cccccc} \mathrm{I} & n_0 & & n_2 & & \dots & \\ \hline \mathrm{II} & & n_1 & & n_3 & & \dots \end{array} \quad \text{for } n_k \in \omega \text{ for all } k \in \omega.$$

We say player I wins the game iff $(n_k)_{k \in \omega} \in A$. Otherwise player II wins. We say $G(A)$ (or A itself) is *determined* iff one of the players has a winning strategy (in the obvious sense).

Which games are determined and what is it good for?

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Determinacy implies regularity properties.

Theorem (Mycielski, Swierczkowski, Mazur, Davis)

If all sets of reals are determined, then all sets of reals are Lebesgue measurable, have the Baire property, and have the perfect set property.

Determinacy for Definable Sets of Reals

Theorem (Martin, 1975)

Assume ZFC. *Then every Borel set of reals is determined.*

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Theorem (Martin-Steel, 1985)

Assume ZFC and there are n Woodin cardinals with a measurable cardinal above them all. Then every $\mathbf{\Sigma}^1_{n+1}$ set is determined.

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Are large cardinals necessary for the determinacy of these sets of reals?
How can these large cardinals affect what happens with the sets of reals?

Iterable Models and Projective Determinacy

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Theorem (Neeman, Woodin)

Let $n \geq 1$. Then the following are equivalent.

(a) $\mathbf{\Sigma}^1_{n+1}$*-determinacy.*

(b) *For every $x \in \mathbb{R}$ the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^\#(x)$ exists.*

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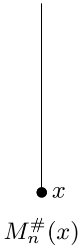

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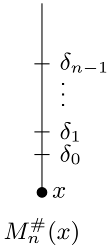

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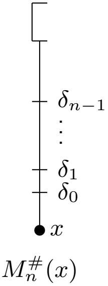

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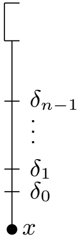

$M_n^\#(x)$

For (a) $\Rightarrow$ (b) see (M, Schindler, Woodin) "Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses", JML 2020.

For (b) $\Rightarrow$ (a) see (Neeman) "Optimal proofs of determinacy II", JML 2002.

Longer Games

Why stop playing at ω?

Longer Games

Why stop playing at ω? Define more generally:

Definition (Gale-Stewart 1953)

Let $A \subset \omega^\alpha$ for some ordinal α. We denote the following game by $G_\alpha(A)$

I	n_0		n_2		$\dots$		n_ω		$\dots$	
II		n_1		n_3		$\dots$		$n_{\omega+1}$		$\dots$

for $n_\beta \in \omega$ for all $\beta < \alpha$.

As before, we say player I wins the game iff $(n_\beta)_{\beta<\alpha} \in A$. Otherwise player II wins. Moreover, $G_\alpha(A)$ (or A itself) is *determined* iff one of the players has a winning strategy (in the obvious sense).

Write $\mathrm{Det}_\alpha(\Lambda)$ for the statement "all games of length α with payoff in Λ are determined".

Obstacles and Observations

Theorem (Mycielski, 1964)

AD_{ω_1}, *determinacy for arbitrary games of length* ω_1, *is inconsistent.*

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Proposition

$\mathrm{Det}_{\omega\cdot(n+1)}(\mathbf{\Pi}^1_1)$ implies $\mathrm{Det}_{\omega}(\mathbf{\Pi}^1_{n+1})$.

Idea: "Simulate" projections by ω moves in a longer game, where we only consider the moves of one of the two players.

Theorem (Neeman, 2004)

Let $\alpha > 1$ *be a countable ordinal and suppose that there are* $-1+\alpha$ *Woodin cardinals with a measurable cardinal above them all. Then* $\mathrm{Det}_{\omega\cdot\alpha}(\mathbf{\Pi}^1_1)$ *holds.*

Determinacy and Large Cardinals

Theorem (Neeman, 2004)

Let $\alpha > 1$ be a countable ordinal and suppose that there are $-1 + \alpha$ Woodin cardinals with a measurable cardinal above them all. Then $\mathrm{Det}_{\omega \cdot \alpha}(\mathbf{\Pi}^1_1)$ holds.

Is this result optimal?

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Is this result optimal?

Let's focus on the first interesting level $\alpha = \omega + 1$.

Theorem (Aguilera-M, JSL 2020)

Suppose $\mathrm{Det}_{\omega\cdot(\omega+1)}(\mathbf{\Pi}^1_1)$. Then there is a premouse with $\omega + 1$ Woodin cardinals.

In fact, the proof only uses $\mathrm{Det}_{\omega^2}(\mathbf{\Pi}^1_2)$.

Larger Countable Ordinals

Theorem (Trang, 2013, building on Woodin)

Let α be a countable ordinal and suppose $\mathrm{Det}_{\omega^{1+\alpha}}(\boldsymbol{\Pi}^1_1)$. Then there is a premouse with ω^α Woodin cardinals.

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Theorem (M, 2019)

Let α be a countable ordinal and suppose $\mathrm{Det}_{\omega^{1+\alpha}}(\mathbf{\Pi}^1_{n+1})$. Then there is a premouse with $\omega^\alpha + n$ Woodin cardinals.

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These methods should allow an analysis of the large cardinal strength of all analytic games of fixed countable length.

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Looking beyond

This part is joint work with Yair Hayut.

κ-Trees

Definition

Let κ be a regular cardinal. A tree T of height κ is called a *normal κ-tree* if

- each level of T has size $<\kappa$,
- every node splits,
- for every $t \in T$ and $\alpha < \kappa$ above the height of t, there is some t' of level α in T such that $t <_T t'$, and
- for every limit ordinal $\alpha < \kappa$ and every branch up to α there is at most one least upper bound in T.

The Branch Spectrum

Definition

Let κ be a regular cardinal. The *Branch Spectrum* of κ is the set

$$\mathfrak{S}_\kappa = \{|[T]| \mid T \text{ is a normal } \kappa\text{-tree}\}.$$

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Examples

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- For $\kappa > \omega$, there are no κ-Kurepa trees iff $\max(\mathfrak{S}_\kappa) = \kappa$.
- For $\kappa > \aleph_1$ of uncountable cofinalty, tree property holds at κ iff $\min(\mathfrak{S}_\kappa) = \kappa$.

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The following gives an upper bound.

Proposition

Let κ be $<\mu$-supercompact, where μ is strongly inaccessible. Then, there is a forcing extension in which κ is weakly compact, $\mathfrak{S}_\kappa = \{\kappa, \kappa^{++}\}$.

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Question

Is this optimal?

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Fact (essentially Solovay)

If $0^\#$ does not exists then every weakly compact cardinal carries a tree with κ^+ many branches.

A Non-domestic Mouse

Theorem (Hayut, M.)

Let κ *be a weakly compact cardinal and let us assume that there is no* κ*-tree with exactly* κ^+ *many branches. Then there is a non-domestic mouse. In particular, there is a model of* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$.

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- Consider a tree $\mathbb{T}(\mathcal{S})$ in $\mathcal{S} = \mathcal{S}(\kappa)$ the stack of mice on $K^c||\kappa$ (cf. Andretta-Neeman-Steel and Jensen-Schimmerling-Schindler-Steel).

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- $\mathbb{T}(\mathcal{S})$ has exactly $(\kappa^+)^V$ many branches (using covering as in JSSS).

A conjecture

What is the large cardinal strength of this statement?

Conjecture

Let κ be a weakly compact cardinal such that there is no κ-tree with exactly κ^+ many branches. Then there is an inner model with a pair of cardinals $\lambda < \mu$ such that λ is $<\mu$-supercompact and μ is inaccessible.

- Inner Model Theory
- Determinacy
- Combinatorics of κ-Trees
- **Looking beyond**

Obstacles

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- But even if we were able to do that, we currently do not know how to "reach it".

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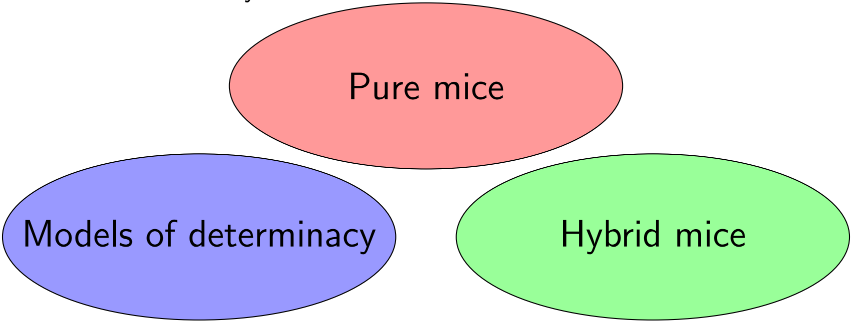

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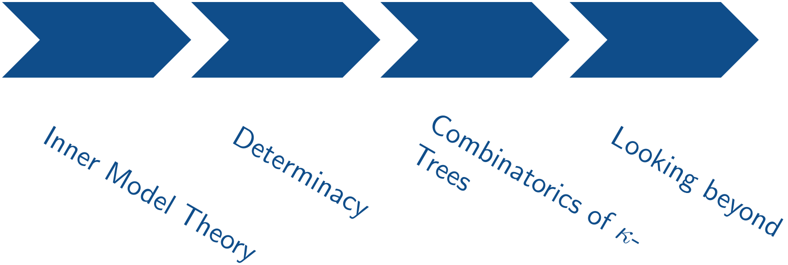
$\rightarrow$ The hierarchy of long games could help here.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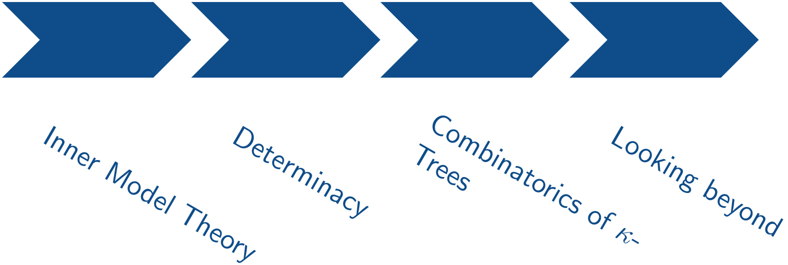

"There is an ever changing list of questions in set theory the answers to which would greatly increase our understanding of the universe of sets. The difficulty of course is the ubiquity of independence: almost always the questions are independent."

(W. H. Woodin in Suitable Extender Models I)