Infinitary proof theory of first order linear logic with fixed points

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Induction and coinduction

Induction	Coinduction - Bisimulation

Induction and coinduction

Termination, Progress; A property eventually holds	Productivity, Equality of streams; A property holds infinitely often
Induction	Coinduction - Bisimulation

Induction and coinduction

Finite data types	Infinite data types
Natural numbers, Lists, etc.	Streams, Infinite trees, etc.
Termination, Progress; A property eventually holds	Productivity, Equality of streams; A property holds infinitely often
Induction	Coinduction - Bisimulation

Induction and coinduction

Least fixed points	Greatest fixed points
Finite data types	Infinite data types
Natural numbers, Lists, etc.	Streams, Infinite trees, etc.
Termination, Progress; A property eventually holds	Productivity,Equality of streams; A property holds infinitely often
Induction	Coinduction - Bisimulation

Mutual least and greatest fixed points

1. Examples?
2. Induction/Coinduction?
3. Termination/productivity?

Prove theorems using induction and coinduction - Previous works

- Induction principle
- Bisimulation
- Coinduction principle [Kozen and Silva]
- An infinitary calculus for first-order logic with inductive definitions [Brotherston]
- A finitary calculus for least and greatest fixed points in linear logic [Baelde]
- Well founded recursion with copatterns and sized types [Abel and Pientka]

Our contribution

A first order calculus for proving properties about mutual least and greatest fixed points, in particular Session-typed processes

1. Add fixed points and assign priorities to them,
2. Use circular edges in the proof for inductive and coinductive steps,
3. Impose a validity condition to ensure soundness of this proof system.

We use priorities in the validity condition to ensure valid simultaneous induction and conduction.

Finite lists: Example of least fixed points

Natural numbers

$$\mathtt{nat} =^1_\mu \oplus\{\mathsf{zero} : 1, \mathsf{succ} : \mathtt{nat}\}$$

$$\overline{3} = \mathsf{succ}\ \mathsf{succ}\ \mathsf{succ}\ \mathsf{zero}$$

Lists of natural numbers

$$\mathtt{list}_{\mathsf{nat}} =^1_\mu \oplus\{\mathsf{nil} : 1, \mathsf{cons} : \mathsf{nat} \otimes \mathtt{list}_{\mathsf{nat}}\}$$

$$\overline{[3, 3]} = \mathsf{cons}(\overline{3}, \mathsf{cons}(\overline{3}, \mathsf{nil}))$$

Programming with finite lists

Terminating

Append two lists

$$
\begin{aligned}
& l \leftarrow \texttt{Append} \leftarrow l_1, l_2 = \\
& \qquad \textbf{case}\, l_1\, (\mu_{list} \Rightarrow \\
& \qquad\quad \textbf{case}\, l_1\, (nil \Rightarrow \textbf{wait}\, l_1; l \leftarrow l_2 \\
& \qquad\qquad cons \Rightarrow l.\mu_{list}; \\
& \qquad\qquad\quad x \leftarrow \textbf{recv}\, l; \\
& \qquad\qquad\quad l.cons; \textbf{send}\, l\, x; l \leftarrow \texttt{Append} \leftarrow l_1\, l_2))
\end{aligned}
$$

If l1 is an empty list (nil): forward l2 to l.

If l1=cons(x, --):
send x to l and call Append
on l2 and the remaining of l1.

I use linear binary session typed processes for programming examples. See [1,2] for more info.

Termination and List as first order predicates

$$List(l_1) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1 \; \_)$$

$$Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil} \; \_) =^1_\mu 1$$
$$Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l_1') \; \_) =^1_\mu Terminate(\_ \leftarrow \text{Append} \leftarrow l_1' \; \_)$$

$$List(\texttt{nil}) =^1_\mu 1$$
$$List(\text{cons}(x) :: l_1') =^1_\mu List(l_1')$$

Append terminates - proof

$List(l_1) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1\ \_)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash 1}\ 1R}{1 \vdash 1}\ 1L}{1 \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil}\ \_)}\ \mu R}{\dagger\ \ List(\texttt{nil}) \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow \texttt{nil}\ \_)}\ \mu L$$

$$\dfrac{\dfrac{\dfrac{\dagger}{List(l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow l_1'\ \_)}}{List(l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l_1')\ \_)}\ \mu R}{\dagger\ \ List(\text{cons}(x) :: l_1') \vdash Terminate(\_ \leftarrow \text{Append} \leftarrow (\text{cons}(x) :: l_1')\ \_)}\ \mu L$$

Programming with streams: example of greatest fixed points

Productive

merge Merge two streams into a single stream by alternatively outputting an element of each.

split_1 Return the odd elements of a stream.

split_2 Return the even elements of a stream.

$$\mathsf{merge}(\mathsf{split}_1(t), \mathsf{split}_2(t)) = t$$

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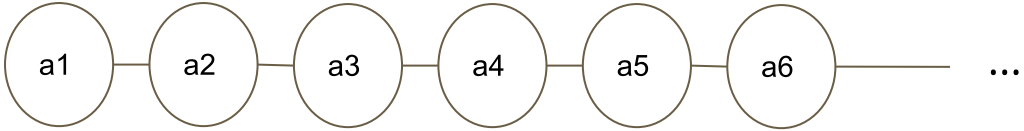

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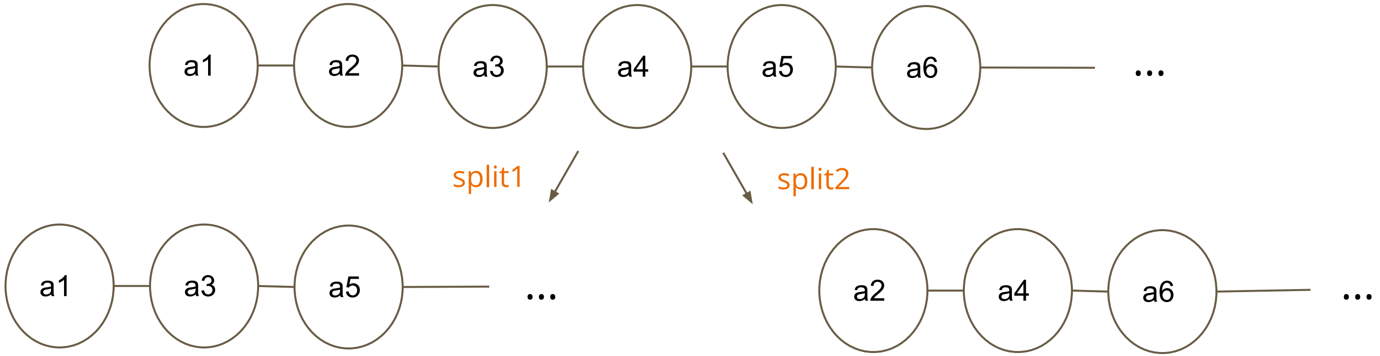

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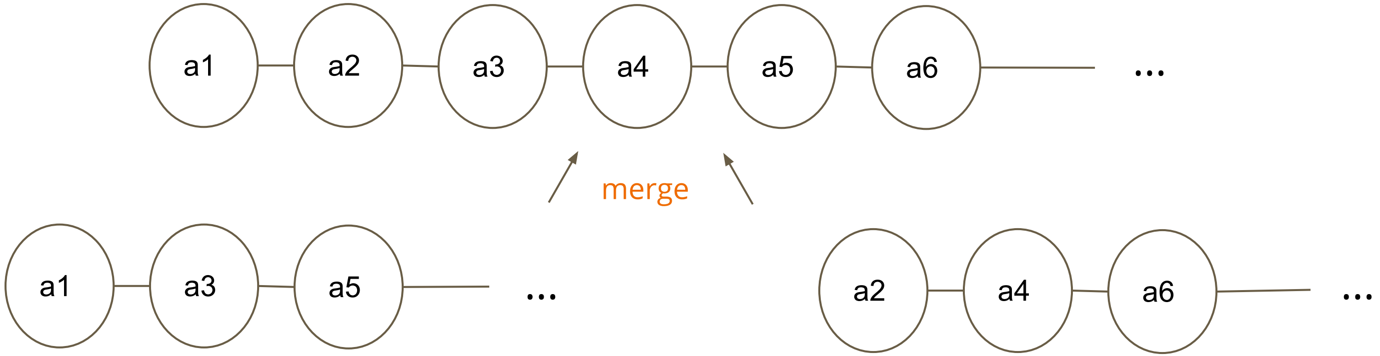

Programming with streams

Define properties of merge and splits as:

$$
\begin{aligned}
\mathtt{Merge}(x, y, z) &=^1_\nu (\mathsf{hd}\, z = \mathsf{hd}\, x \,\&\, \mathtt{Merge}\,(y, \mathsf{tl}\, x, \mathsf{tl}\, z)) \\
\mathtt{Split}_1(x, y) &=^1_\nu (\mathsf{hd}\, y = \mathsf{hd}\, x \,\&\, \mathtt{Split}_2(\mathsf{tl}\, x, \mathsf{tl}\, y)) \\
\mathtt{Split}_2(x, y) &=^1_\nu (1 \,\&\, \mathtt{Split}_1(\mathsf{tl}\, x, y))
\end{aligned}
$$

Operations merge and split□ are inverses

$$\frac{}{\cdot \vdash \mathsf{hd}\,y_1 = \mathsf{hd}\,y_1}\;=R$$

$$\frac{\cdot \vdash \mathsf{hd}\,y_1 = \mathsf{hd}\,y_1}{\mathsf{hd}\,y_1 = \mathsf{hd}\,x \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1}\;=L$$

$$\frac{\mathsf{hd}\,y_1 = \mathsf{hd}\,x \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1}{\mathsf{hd}\,y_1 = \mathsf{hd}\,x, 1 \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1}\;1L$$

$$\frac{\mathsf{hd}\,y_1 = \mathsf{hd}\,x, 1 \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1}{\mathsf{hd}\,y_1 = \mathsf{hd}\,x, 1\&\mathtt{S}_1(\mathtt{tl}\,x, y_2) \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1}\;\&L$$

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$$\frac{\mathtt{S}_2(x, y_2), \mathtt{S}_1(x, y_1) \vdash \mathtt{M}(y_1, y_2, x)}{\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), \mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}\;\mathsf{Sub}_{[\mathsf{tl}x, \mathsf{tl}y_1, y_2 / x, y_2, y_1]}$$

$$\frac{\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), \mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}{\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), 1\&\mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}\;\&L$$

$$\frac{\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), 1\&\mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}{\mathsf{hd}y_1 = \mathsf{hd}x\&\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), 1\&\mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}\;\&L$$

$$\frac{\mathsf{hd}y_1 = \mathsf{hd}x\&\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), 1\&\mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathsf{hd}x = \mathsf{hd}y_1 \qquad \mathsf{hd}y_1 = \mathsf{hd}x\&\mathtt{S}_2(\mathsf{tl}x, \mathsf{tl}y_1), 1\&\mathtt{S}_1(\mathtt{tl}x, y_2) \vdash \mathtt{M}(y_2, \mathsf{tl}y_1, \mathsf{tl}x)}{\mathsf{hd}\,y_1 = \mathsf{hd}\,x \,\&\, \mathtt{S}_2(\mathsf{tl}\,x, \mathsf{tl}\,y_1), 1\&\mathtt{S}_1(\mathtt{tl}\,x, y_2) \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\,y_1, \mathsf{tl}\,x)}\;\&R$$

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$$\frac{\mathsf{hd}\,y_1 = \mathsf{hd}\,x \,\&\, \mathtt{S}_2(\mathsf{tl}\,x, \mathsf{tl}\,y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\,y_1, \mathsf{tl}\,x)}{\mathtt{S}_1(x, y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\,y_1, \mathsf{tl}\,x)}\;\nu L$$

$$\frac{\mathtt{S}_1(x, y_1), \mathtt{S}_2(x, y_2) \vdash \mathsf{hd}\,x = \mathsf{hd}\,y_1 \,\&\, \mathtt{M}(y_2, \mathsf{tl}\,y_1, \mathsf{tl}\,x)}{\mathtt{S}_1(x, y_1), \mathtt{S}_2(x, y_2) \vdash \mathtt{M}(y_1, y_2, x)}\;\nu R$$

Programming with mutual least and greatest fixed points

run(x,t): A stream producer where x is the list of operations, and t is the output stream.

Skip one step and do nothing

Put z as the head of output stream and inserts the new list of operations x to the original one.

$$
\begin{aligned}
\mathtt{run}(\cdot\,, t) &=^1_\mu \quad 1 \\
\mathtt{run}(skip; x, t) &=^1_\mu \quad \mathtt{run}(x, t) \\
\mathtt{run}(put\langle x \rangle; y, t) &=^1_\mu \quad \mathtt{nrun}\,(x, y, t) \\
\mathtt{nrun}(x, y, t) &=^2_\nu \quad \mathtt{hd}\, t = \mathtt{o} \,\&\, \mathtt{run}(x; y, \mathtt{tl}\ t)
\end{aligned}
$$

Run on any list of operations produces a (possibly infinite) list of elements "o"

$$\begin{aligned}
\mathtt{run}(\cdot\,, t) &=^1_\mu 1 \\
\mathtt{run}(skip; x, t) &=^1_\mu \mathtt{run}(x, t) \\
\mathtt{run}(put\langle x\rangle; y, t) &=^1_\mu \mathtt{nrun}\,(x, y, t) \\
\mathtt{nrun}(x, y, t) &=^2_\nu \mathtt{hd}\, t = \mathtt{o} \,\&\, \mathtt{run}(x; y, \mathtt{tl}\, t)
\end{aligned}$$

$$\begin{aligned}
\mathtt{list}_\mathtt{o}(t) &=^1_\mu \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_\mathtt{o}(t)\} \\
\mathtt{stream}_\mathtt{o}(t) &=^2_\nu \&\{\mathsf{hd} : \mathtt{hd}\, t = \mathtt{o}, \mathsf{tl} : \mathtt{list}_\mathtt{o}\,(\mathtt{tl}\, t)\}
\end{aligned}$$

$$(\dagger)\ \mathtt{run}(x, t) \vdash \mathtt{list}_\mathtt{o}(t)$$

$$(\star)\ \mathtt{nrun}(x, y, t) \vdash \mathtt{stream}_\mathtt{o}(t)$$

Run produces a list$_\circ$ - proof

$$
\begin{aligned}
\mathtt{run}(\cdot\,, t) &=^1_\mu 1\\
\mathtt{run}(\mathit{skip}; x, t) &=^1_\mu \mathtt{run}(x, t)\\
\mathtt{run}(\mathit{put}\langle x\rangle; y, t) &=^1_\mu \mathtt{nrun}\,(x, y, t)\\
\mathtt{nrun}(x, y, t) &=^2_\nu \mathsf{hd}\, t = \mathsf{o} \,\&\, \mathtt{run}(x; y, \mathtt{tl}\, t)
\end{aligned}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash 1}\;1R}{\cdot \vdash \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_\circ(t)\}}\;\oplus R}{\cdot \vdash \mathtt{list}_\circ(t)}\;\mu_{\mathtt{list}_\circ} R}{1 \vdash \mathtt{list}_\circ(t)}\;1L}{\dagger\, \mathtt{run}(\cdot\,, t) \vdash \mathtt{list}_\circ(t)}\;\mu_{\mathtt{run}} L
$$

$$
\dfrac{\dfrac{\dagger}{\mathtt{run}(x, t) \vdash \mathtt{list}_\circ(t)}}{\dagger\, \mathtt{run}((\mathit{skip}; x), t) \vdash \mathtt{list}_\circ(t)}\;\mu_{\mathtt{run}} L
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\star}{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{stream}_\circ(t)}}{\mathtt{nrun}\,(x, y, t) \vdash \oplus\{\mathsf{nil} : 1, \mathsf{next} : \mathtt{stream}_\circ(t)\}}\;\oplus R}{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{list}_\circ(t)}\;\mu_{\mathtt{list}_\circ} R}{\dagger\, \mathtt{run}(\mathit{put}\langle x\rangle; y, t) \vdash \mathtt{list}_\circ(t)}\;\mu_{\mathtt{run}} L
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{hd}\, t = \mathsf{o} \vdash \mathsf{hd}\, t = \mathsf{o}}\;\mathtt{ID}}{\&\{\mathsf{hd} : \mathsf{hd}\, t = \mathsf{o}, \mathsf{tl} : \mathtt{run}(x; y, \mathtt{tl}\, t)\} \vdash \mathsf{hd}\, t = \mathsf{o}}\;\&L}{\mathtt{nrun}\,(x, y, t) \vdash \mathsf{hd}\, t = \mathsf{o}}\;\nu_{\mathtt{nrun}} L \qquad \dfrac{\dfrac{\dfrac{\dagger}{\mathtt{run}(x; y, \mathtt{tl}\, t) \vdash \mathtt{list}_\circ(\mathsf{tl}\, t)}}{\&\{\mathsf{hd} : \mathsf{hd}\, t = \mathsf{o}, \mathsf{tl} : \mathtt{run}(x; y, \mathtt{tl}\, t)\} \vdash \mathtt{list}_\circ(\mathsf{tl}\, t)}\;\&L}{\mathtt{nrun}\,(x, y, t) \vdash \mathtt{list}_\circ(\mathsf{tl}\, t)}\;\nu_{\mathtt{nrun}} L}{\dfrac{\mathtt{nrun}\,(x, y, t) \vdash \&\{\mathsf{hd} : \mathsf{hd}\, t = \mathsf{o},\ \mathsf{tl} : \mathtt{list}_\circ(\mathsf{tl}\, t)\}}{\star\, \mathtt{nrun}\,(x, y, t) \vdash \mathtt{stream}_\circ(t)}\;\nu_{\mathtt{stream}_\circ} R}\;\&R
$$

Strong progress and Validity condition

A process satisfies *strong progress*, if after *finite number of steps*, it either becomes *empty* or attempts to *communicate to the left or right* [2].

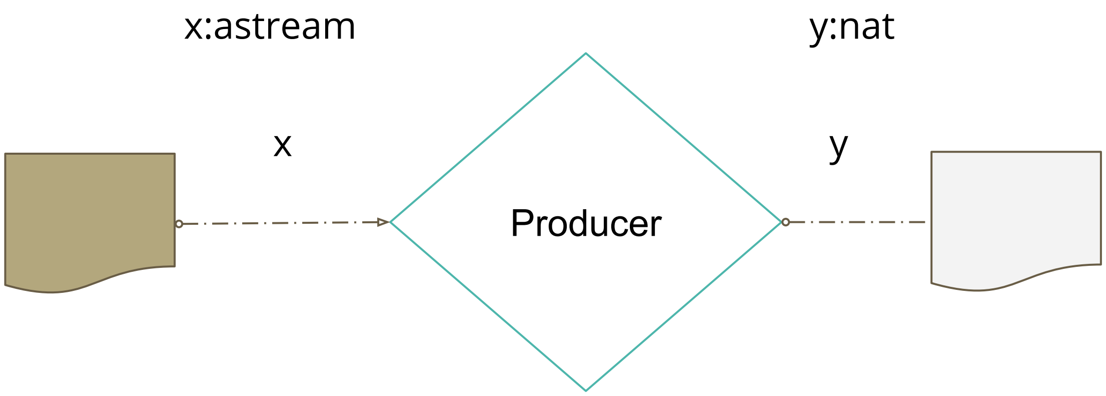
Theorem. Our *validity condition* on session-typed processes ensures *strong progress* [2].

We want to prove this directly using our calculus.

Producer/Idle: a locally valid program

$$\begin{aligned}\Sigma := \mathsf{ack} &=^1_\mu \oplus\{ack : \mathsf{astream}\},\\ \mathsf{astream} &=^2_\nu \&\{head : \mathsf{ack},\quad tail : \mathsf{astream}\},\\ \mathsf{nat} &=^3_\mu \oplus\{z : 1,\quad s : nat\}\end{aligned}$$

$$z : \mathsf{ack} \vdash w \leftarrow \mathtt{Idle} \leftarrow z :: (w : \mathsf{nat})$$
$$x : \mathsf{astream} \vdash y \leftarrow \mathtt{Producer} \leftarrow x :: (y : \mathsf{nat}),$$



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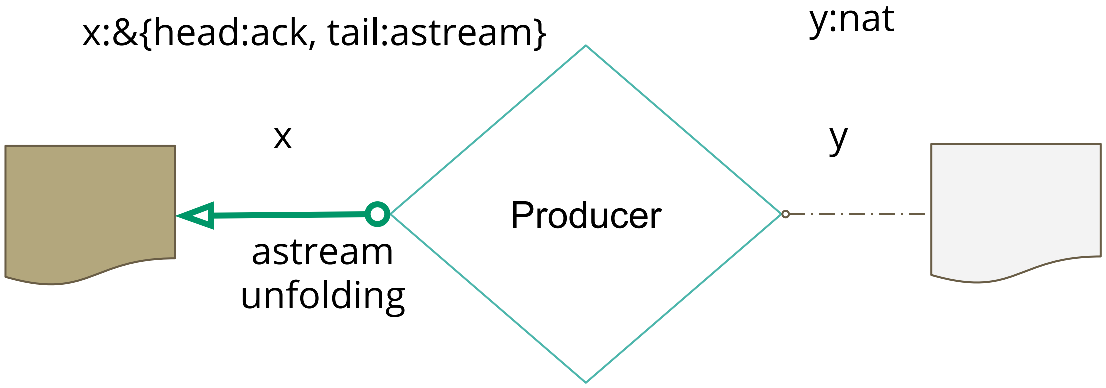

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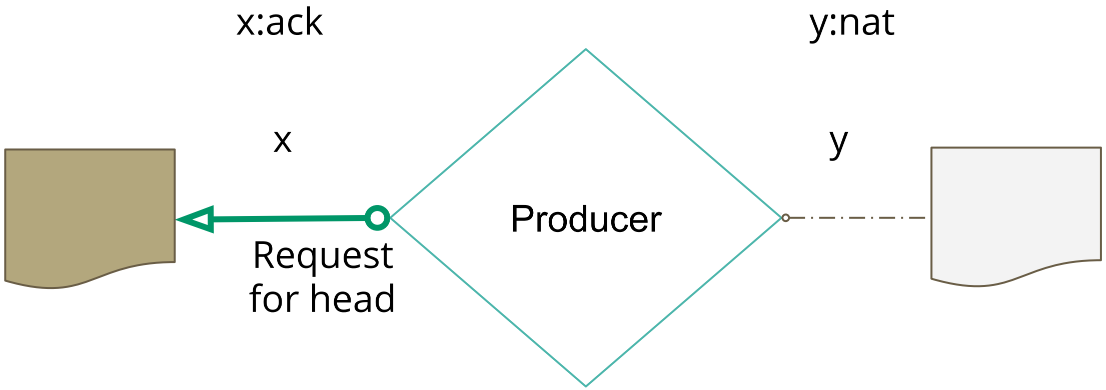

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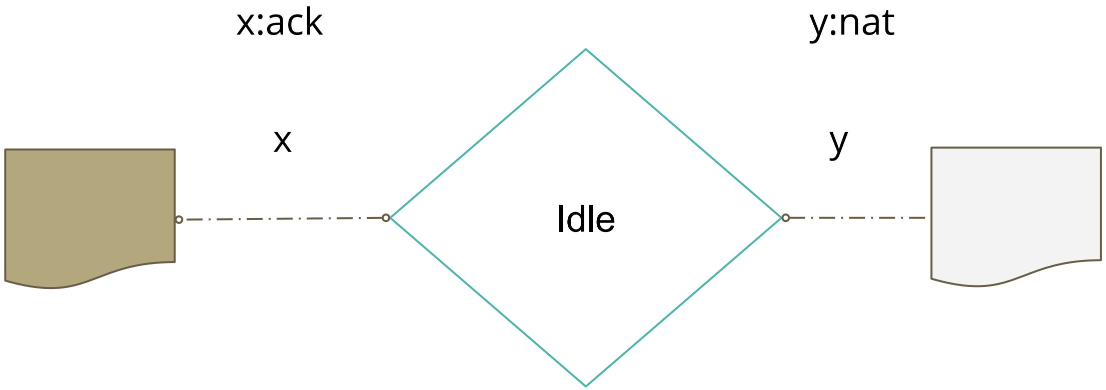

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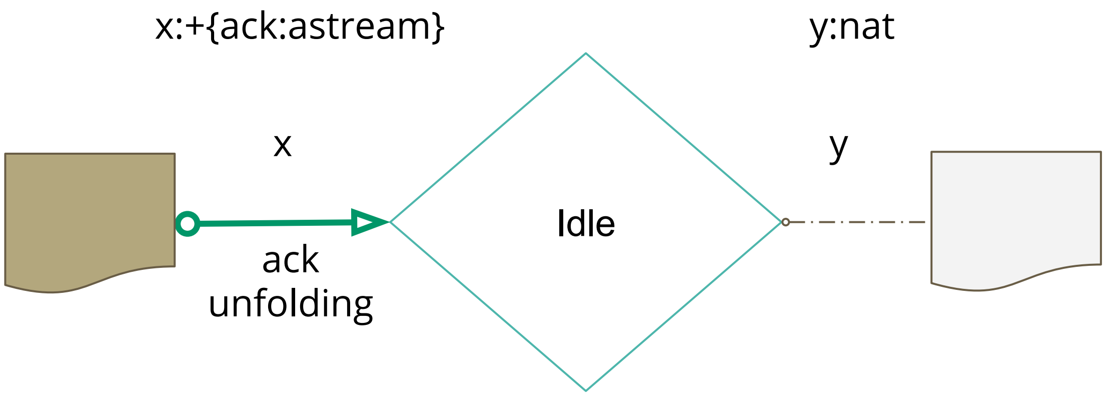

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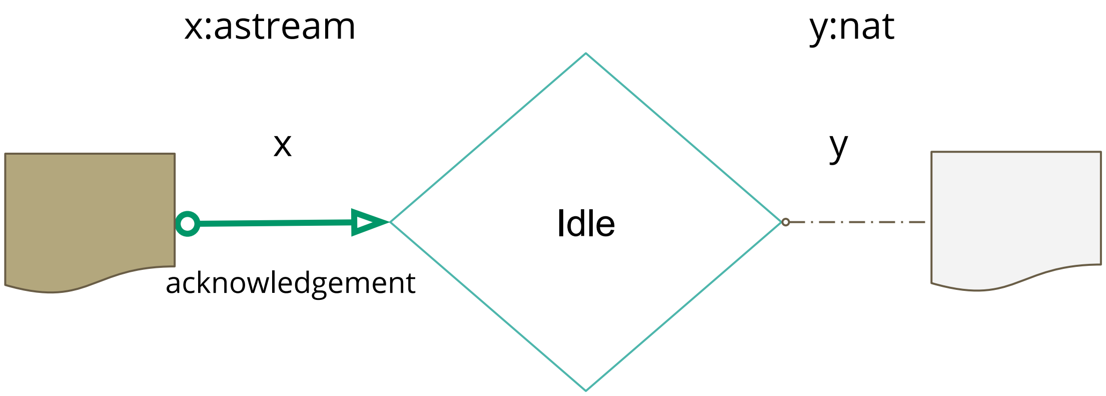

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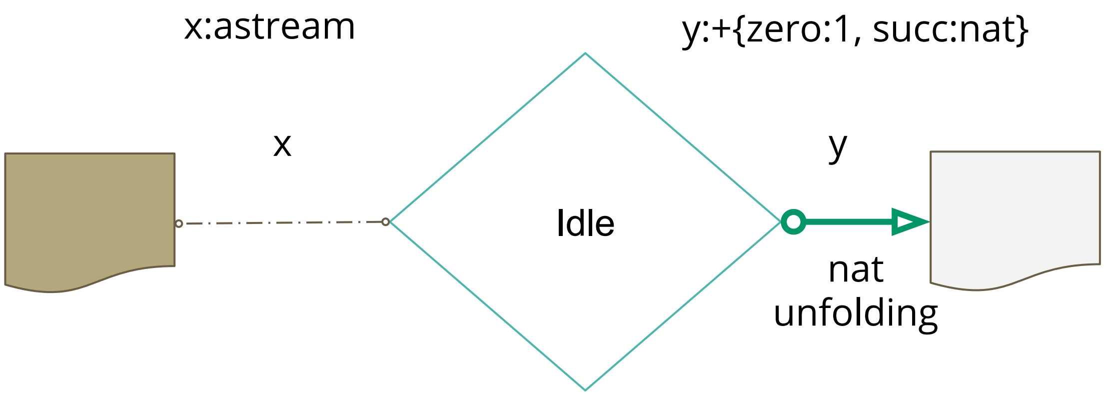

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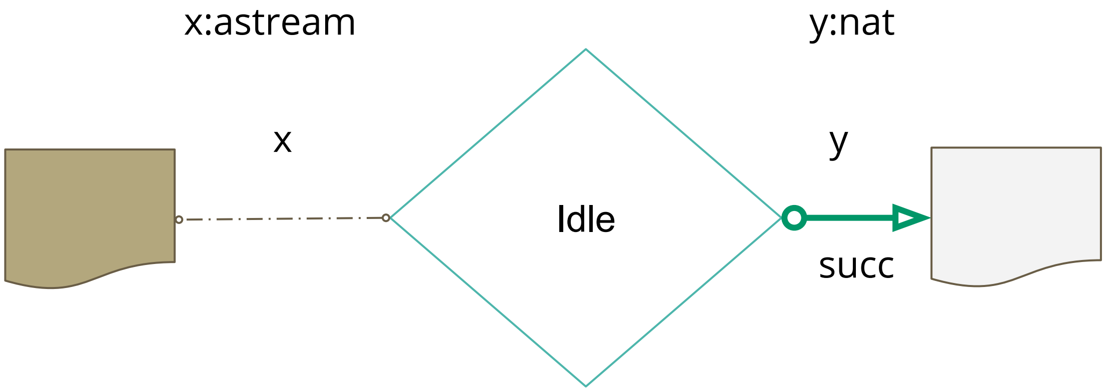

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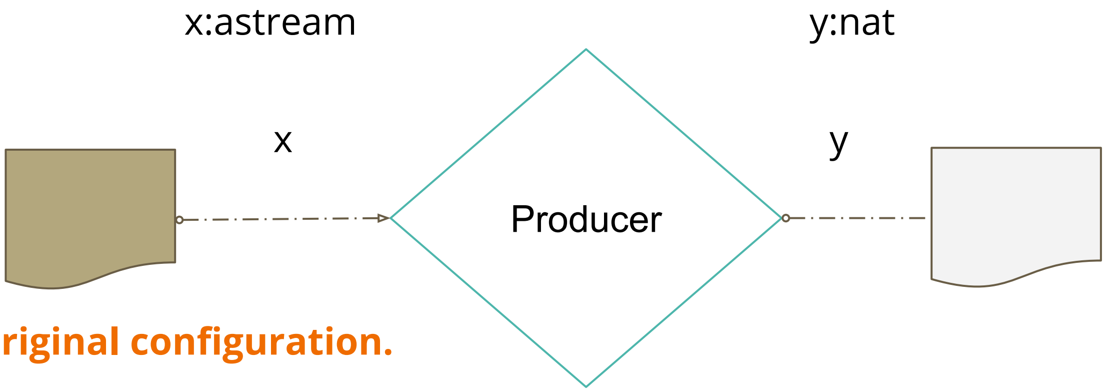

Back to the original configuration.

Producer/Idle: a locally valid program - code

$$\Sigma := \text{ack} =^1_\mu \oplus\{ack : \text{astream}\},$$
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$$\text{nat} =^3_\mu \oplus\{z : 1, \quad s : nat\}$$

Eventually communicate with its external channels

$z : \text{ack} \vdash w \leftarrow \texttt{Idle} \leftarrow z :: (w : \text{nat})$
$x : \text{astream} \vdash y \leftarrow \texttt{Producer} \leftarrow x :: (y : \text{nat}),$

$$y^0 \leftarrow \texttt{Producer} \leftarrow x^0 = \\ \quad Lx^0.\nu_{astream}; \\ \quad\quad Lx^1.head; y^0 \leftarrow \texttt{Idle} \leftarrow x^1$$

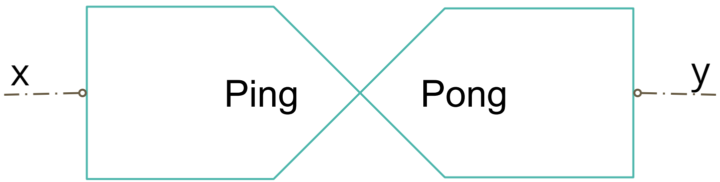
$$y^0 \leftarrow \texttt{Idle} \leftarrow x^1 = \\ \quad \mathbf{case}\ Lx^1\ (\mu_{ack} \Rightarrow \\ \quad\quad \mathbf{case}\ Lx^2\ (ack \Rightarrow Ry^0.\mu_{nat}; \\ \quad\quad\quad Ry^1.s; y^1 \leftarrow \texttt{Producer} \leftarrow x^2))$$

This example is adapted from [2].

Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack},\quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1,\quad s : nat\}$$

$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack},\quad tail : \mathsf{astream}\},$$
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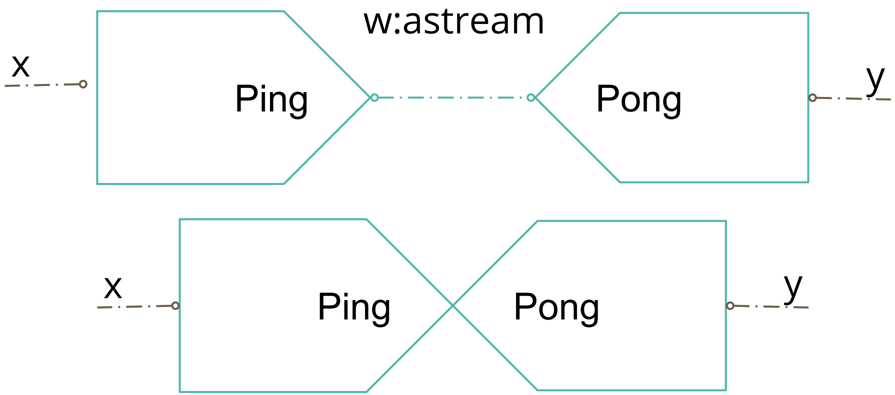
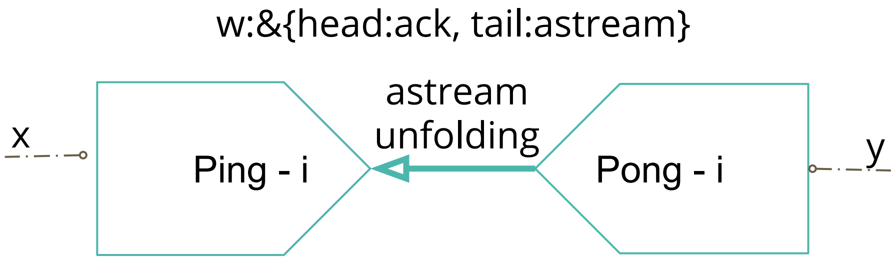
$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1, \quad s : nat\}$$

$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
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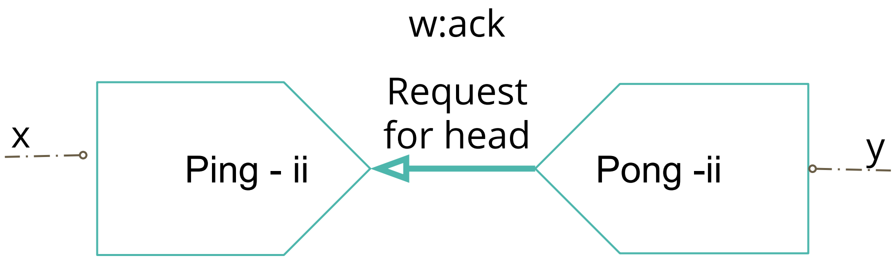
$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
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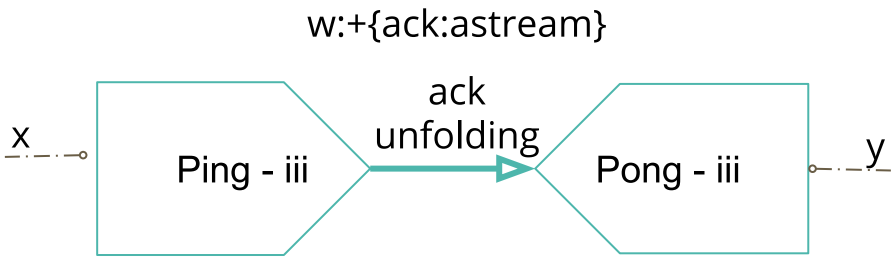
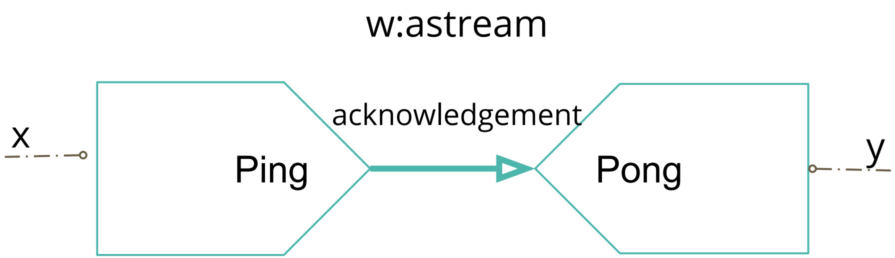
$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Ping-Pong: an invalid program

$$\Sigma := \mathsf{ack} =^1_\mu \oplus\{ack : \mathsf{astream}\},$$
$$\mathsf{astream} =^2_\nu \&\{head : \mathsf{ack}, \quad tail : \mathsf{astream}\},$$
$$\mathsf{nat} =^3_\mu \oplus\{z : 1, \quad s : nat\}$$

$$x : \mathsf{nat} \vdash \mathtt{Ping} :: (w : \mathsf{astream})$$
$$w : \mathsf{astream} \vdash \mathtt{Pong} :: (y : \mathsf{nat})$$
$$x : \mathsf{nat} \vdash \mathtt{PingPong} :: (y : \mathsf{nat})$$



Back to the original configuration.

Ping-Pong: an invalid program - code

$y \leftarrow \texttt{PingPong} \leftarrow x =$

$\quad w \leftarrow \texttt{Ping} \leftarrow x;$ $\qquad \%\ \textit{spawn process}\ \texttt{Ping}_w$

$\quad\quad y \leftarrow \texttt{Pong} \leftarrow w$ $\qquad \%\ \textit{continue with a tail call}$

Keep calling itself without communicating with its external channels

$w \leftarrow \texttt{Ping} \leftarrow x =$	$[0,0,0,0,0,0]$
$\quad \textbf{case}\ Rw\ (\nu_{astream} \Rightarrow$	$[0,0,-1,0,0,0]$
$\quad\quad \textbf{case}\ Rw\ (head \Rightarrow Rw.\mu_{ack};$	$[0,1,-1,0,0,0]$
$\quad\quad\quad Rw.ack; w \leftarrow \texttt{Ping} \leftarrow x$	$[0,1,-1,0,0,0]$
$\quad\quad \mid tail \Rightarrow w \leftarrow \texttt{Ping} \leftarrow x))$	$[0,0,-1,0,0,0]$

$y \leftarrow \texttt{Pong} \leftarrow w =$	$[0,0,0,0,0,0]$
$\quad Lw.\nu_{astream};$	$[0,0,0,1,0,0]$
$\quad Lw.head;$	$[0,0,0,1,0,0]$
$\quad \textbf{case}\ Lw\ (\mu_{ack} \Rightarrow$	$[-1,0,0,1,0,0]$
$\quad\quad \textbf{case}\ Lw\ ($	$[-1,0,0,1,0,0]$
$\quad\quad\quad ack \Rightarrow Ry.\mu_{nat};$	$[-1,0,0,1,0,1]$
$\quad\quad\quad Ry.s;$	$[-1,0,0,1,0,1]$
$\quad\quad\quad y \leftarrow \texttt{Pong} \leftarrow w))$	$[-1,0,0,1,0,1]$

A valid configuration of processes satisfies strong progress

We define strong progress as a predicate

$$\mathcal{C} \in [\![x : A]\!]$$

$$\cdot \vdash \mathcal{C} :: (x{:}A) \quad \Longleftrightarrow_{\text{Bisimulation}} \quad \cdot \vdash \mathcal{C} \in [\![x : A]\!]$$

Theorem. *If configuration C is well-typed then there is an infinite derivation for its strong progress property. Moreover, if it C is valid, the infinite derivation is a proof.*

Conclusion

We introduced an infinitary sequent calculus for first order intuitionistic multiplicative additive linear logic with fixed points [2].

Our main motivation for introducing this calculus is to reason about programs behaviour. In particular we use this calculus to give a direct proof for the strong progress property of locally valid binary session typed processes [2]. The importance of a direct proof other than its elegance is that it can be adapted for a more general validity condition on processes without the need to prove cut elimination productivity for their underlying derivations.

Send me an Email!

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