How the Law of Excluded Middle Pertains to the Second Incompleteness Theorem and its Boundary-Case Exceptions

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Talk was also scheduled to appear in ASL 2020 Conference,
before Covid-19 forced conference cancellation

1. Slides 1-15 were presented at LFCS 2020 conference, meeting during Janaury 4-7, 2020, in Deerfield Florida (under a technically different title)
2. Published as refereed 19-page paper in Springer's LNCS 11972, pp. 268-286 in Springer's "Proceedings of Logical Foundations of Computer Science 2020"
3. Second presentaion of this talk was intended for ASL-2020 conference, before Covid-19 cancelled conference.

Willard recommends you read Item 2's 19-page paper after skimming these slides.

- They nicely summarize my six JSL and APAL papers.
- Manuscript is also more informative than the sketch outlined here.

Manuscript helps one understand all of Willard's papers

1. Main Question left UNANSWERED by Gödel:

2nd Incomplet. Theorem indicates strong formalisms cannot verify their own consistency.

..... *But Humans Presume Their Own Consistency,*

at least PARTIALLY, as a prerequisite for Thinking ?

A Central Question: *What systems* are ADEQUATELY WEAK to hold **some type (?)** *knowledge their own consistency?*

- *ONLY PARTIAL ANSWERS* to this Question are Feasible

  because 2nd Inc Theorem is QUITE STRONG !

Let LXM = Law of Excluded Middle

- OurTheme: Different USES of LXM do change Breadth and LIMITATIONS of 2nd Inc Theorem

1. Hilbert NEVER WITHDREW plan to *repair* his program.
2. Gödel's seminal 1931 paper states 2nd Inc Theorem
   "represents no contradiction of (main) standpoint of Hilbert."
3. Gerald Sacks (YouTube Talk at UPenn) reports Gödel saying Hilbert's Consistency Program was "VERY MUCH ALIVE" and "INTERESTING".

Let LXM = Law of Excluded Middle

- OurTheme: *Delicate* USES of LXM can change Breadth and LIMITATIONS of 2nd Inc Theorem

E.G. Treating LXM Schema as Set of Theorems (instead of as class of axioms) CAN EXPLAIN Hilbert's + Gödel's Ambivalence in 1-3.

2b. More Precise Summary of Our Goals:

Again, let LXM = Law of Excluded Middle

1. TAB = Semantic Tableaux Deduction
2. XTAB = Expanded Version of TAB where LXM treated as Logical Axioms (instead of as theorems)

Our MAIN THEME: Systems Owning Some Appreciation of their Self-Consistency are Feasible Under TAB BUT NOT XTAB deduction.

- Because of XTAB's **EXCESSIVE USE** of LXM.

CENTRAL POINT; An **Unusual** *(weaker)* **Use of LXM** (under "TAB") helps acheive goals of Hilbert + Gödel.

3. 3-PART Notation Germane to Central Formalism :

Def: Let Add(x, y, z) and Mult(x, y, z) be two atomic predicates. Then axiom system α called:

- **Type-A** iff it contains Equation 1 as axiom:
- **Type-M** iff contains Equations 1 + 2 as axioms

$$\forall x \ \forall y \ \exists z \quad Add(x, y, z) \tag{1}$$

$$\forall x \ \forall y \ \exists z \quad Mult(x, y, z) \tag{2}$$

- **Type-S** iff it views only Successor as "Total" Function.

Above Notation Germane to Many Generalizations and Evasions of 2nd Inc Theorem, including:

1. Many Type-A evade 2nd Inc under TAB deduction
2. Most Type-M obey 2nd Inc under TAB.
3. Most Type-S ALSO OBEY 2nd Inc under XTAB

4. Summary of Prior Litrature

1. Kreisel-Takeuti (1974) explored Set-Theoretic Exceptions to 2nd Inc Theorem under Gentzen-like cut-free deduction.
2. Pudlák-1985 + Solovay-1994 (using Nelson+Wilkie-Paris) showed 2nd Inc Theorem holds when Succesor is total function under Frege-Hilbert deduction.
3. Willard 1993-2020: Several generalizations and exceptions to 2nd Inc Theoem for *Weak Axiomatic* systems.
4. Artemov-2019: Some pragmatic exceptions to 2nd Inc Theorem where Peano Arithmic can verify its self consistency in a *Fragmentary Sense*.

5. Summary of Our Evasions and Generalizations of 2nd Incomp. Theorem

1. **PRAGMATIC** Evasions of 2nd Inc Theorem under Types A recognizing their consistency under TAB deduction.
2. Similar Evasion of Gödel Theorem **INFEASIBLE** for Types M recognizing their consistency under TAB deduction.
3. BOTH Types S and A are **UNABLE to** recognize their consistency under XTAB deduction.

REMINDER: All Our TYPES treat Add(x,y,z) and Mult(x,y,z) as 3-way relations, where earlier slide noted ....

- Type-A treats only additon as "total" function,
- Type-M treats add + multiply as "total" functions.
- Type-S treats only Successor as "total"

6. Definition of "Self-Justification" :

Axiom System β called **SELF JUSTIFYING RELATIVE** to Deduction Method d when :

1. one of β's formal theorems (or axioms) states d's deduction method, applied to axiom system β, is consistent.
2. and the axiom system β is **also actually** consistent.

Next Two slides Illustrates Example.

7. Comparing Kleene-1938 and Willard's Results

where β called **"SELF JUSTIFYING"** System when :

1. one of β's formal theorems (or axiom) states d's deduction method, applied to axiom system β, is consistent.
2. and the axiom system β **actually is** consistent.

Theorem (Kleene 1938): $\forall\ \alpha\ \ \forall\ d$ Any axiom system α can be extended into broader system $\alpha^d \supseteq \alpha$ satisfying " **PART 1** ".

i.e. Let $\alpha^d = \alpha \cup \text{SelfCons}(\alpha, d)$ (where Blue Line defines latter)

"Exists NO PROOF of $0 = 1$ from α^d via d's deduction method"

Above Well Defined But Catch is α^d Usually Fails " **PART 2** " .

W-'s Main Goal: Show Judiciously Chosen α^d **solves this problem.**

8. 1993 Example of "IS(α)" Self Justifying Arithmetic.

Notation: α is a Base Axiom System (such as Peano Arithmetic)

IS(α) $=_{def}$ Axiom System **with following 3 parts:**

1. Initially defined functions and relations where Addition is Total Function and Multiplication is **ONLY** 3-way relation.

2. Some r.e. encoded set of Π_1 axioms, called "Group-2" schema, capable of proving all α's Π_1 theorems. Example: For each Π_1 sentence Φ , this schema may include axiom:

$$\forall\, p \quad \{ \ \mathrm{Prf}_\alpha(\ulcorner \Phi \urcorner , p) \ \Rightarrow \ \Phi \ \}$$

3. Declaration of IS(α)'s consistency under Sem Tableaux deduction, such as: "No semantic tableaux proof of 0=1 from union of Group 1 + 2 axioms with this sentence (looking at itself)".

9. Main Theorem about IS(α)'s 3-Part Axiom System

Main Theorem (W– 1993,2001,2005,2006,2020)

If α is consistent then IS(α) is consistent

1. This theorem is Subtle because its Analog **FAILS** when either IS(α)'s first axiom includes multiplication as total function OR IS(α)'s third axiom uses Hilbert deduction.
2. Next Slide explains intuition why above theorem allows IS(α) to view Addition BUT NOT MULTIPLICATION as total function.

Compare Sequenes $a_1,\ a_2,\ a_3, \ldots$ and $b_1,\ b_2,\ b_3, \ldots$ where:

- $a_1 \;=\; 2 \;=\; b_1$
- $a_{j+1} \;=\; a_j \;+\; a_j$
- $b_{j+1} \;=\; b_j \;*\; b_j$

Then existence-proofs for a_N and b_N **BOTH HAVE** O(N) Lengths

$$\text{BUT } a_N = 2^N \text{ while } b_N = 2^{2^{N-1}}$$

This distinction explains why IS(α) treats Addition BUT NOT Multiply as a total function when it evades Gödel's Theorem.

- E.G. Object b_N has O(2^N) length, that is **TOO LARGE** to reasonably appear in proof of length $\cong$ N

11. Why XTAB satisfies 2nd Inc Theorem **MORE FULLY** than TAB does.

1. Pudlák-Solovay show Type-S arithmetics **UNABLE** to recognize their Frege-Hilbert Consistency
2. Above because of Hilbert Deduction's "Linear Rule" below:

$$\text{PrfLength}(\Psi) \quad \leq \quad \text{PrfLength}(\Phi) \; + \; \text{PrfLength}(\Phi \Rightarrow \Psi)$$

3. But XTAB **unlike TAB** also satisfies this Linear Rule (because XTAB treats **LXM** phrases as logical axioms).

Thus 1+2+3 **ROUGHLY** Explain How Pudlák-Solovay Tech **GENERALIZES** for XTAB deduction under Type-S Arithm.

- MORE on NEXT TWO Slides.

12. Def. of XTAB proof of theorem Ψ from ax system β

A Smullyan-style tree structure with root $= \neg\Psi$ and where:

- All paths **conclude** with a pair of **contradictory** sentences.
- Internal Nodes are Either β−axioms, instances of LXM, **OR** deductions via Elimination Rules applied to ancestor nodes,

Three Examples of Elimination Rules:

1. $\neg\neg\Upsilon \Longrightarrow \Upsilon$.
2. $\Upsilon \wedge \Gamma \Longrightarrow \Upsilon$ and $\Upsilon \wedge \Gamma \Longrightarrow \Gamma$.
3. A pair of sibling nodes Υ and Γ is allowed when their ancestor is $\Upsilon \vee \Gamma$.

Lemma $\forall\, \phi$ Any node V in XTAB proof may contain ϕ and $\neg\, \phi$ as its *only* **grandchildren**.

13. **More About** Why XTAB satisfies 2nd Inc Theorem, **MORE FULLY** than TAB does.

Lemma $**$ $\forall\ \phi$ Any node V in XTAB proof may contain ϕ and $\neg\ \phi$ as its *only* **grandchildren**.

- This lemma does NOT HOLD also for Tab proofs !

Lemma $**$ is crucial to UNDERSTANDING XTAB because:

1. Subtree below ϕ can store proof of $\phi \ \Rightarrow\ \psi$
2. Subtree below $\neg\ \phi$ can store proof of ϕ

Hence 1 + 2 imply Ψ's proof can satisfy:

$$\text{PrfLength}(\psi) \ \leq\ \text{PrfLength}(\phi) + \text{PrfLength}(\phi \Rightarrow \psi)$$

Thus, we can use analog of Pudlák-Solovay Method to show XTAB STRONGLY SUPPORTS 2nd Inc Theorem.

Self-Justification Theorem (W– 1993,2001,2005,2006,2020)

If α is consistent then IS(α) is consistent

Proof by Contradiction:

1. If IS(α) is inconsistent then MINIMAL-SIZED proof of $0 = 1$ from IS(α) must exist.
2. Such a proof P CANNOT REFERENCE ITSELF or any other proof of $0 = 1$ from IS(α) because of Addition's Slow Growth property.
3. Therefore, P MUST ALSO be Proof of $0 = 1$ FROM α

Latter Observation **FINISHES PROOF** because it contradicts

Initial Assumption that α is consistent.

15. Why Revisiting Hilbert's Consistency Program is Desirable :

1. Gödel **NEVER ADVOCATED** killing H-Program in his seminal 1931 paper.

   He "EXPRESSLY" encouraged H's Continuation.

2. Gerald Sacks (YouTube from UPenn) **RECALLS** Gödel making **ADDITIONAL** comments that Hilbert's Goals **''SHOULD NOT BE ERASED"**.

3. Willard in seven JSL+APAL+LFCS articles showed DILUTED form of H-Program was FEASIBLE when using "I am Consistent" axiom-sentences.

Puzzling Facet of Current Paper: **LXM permissibly treated** as class of "theorems" **BUT NOT** as **"Logical Axioms".**

16. Slightly AMUSING Comment and VERY SERIOUS Conclusion :

1. Remember the proverb: "The book is BETTER than the movie".
2. Thus analogously, you will find quite interesting my paper in Springer's LNCS # 11972, pp. 268-286.
3. Or alternatively save money by downloading a CRUDER and earlier version of these results from Cornell archives at arXiv 1807.04717

A Very Surprising Fact about the Cornell Report :

- Although mostly cruder and less polished than my Springer article, Section 6 of the arXiv report ACTUALLY goes MUCH FURTHER into its special topic.... Thus, it almost (but not quite) anticipates the Epidemic, when it dicusses the Fermi Paradox in some unusual extemperaneous comments that anticipate and speculate about a potentially plausible application of Self-Justifying Logics, that could potentially (?) arise in the very distant future ....

NO similar ANALOGS of Section 6 from arXiv (issued in 2018) appear in LNCS paper about our new interpretation of the Fermi Paradox.