Reduction games, provability, and compactness

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Joint work with Damir D. Dzhafarov and Denis R. Hirschfeldt

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Background on Π^1_2-problems

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We denote such problems by P and Q.

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Theorem (Folklore/Wang)

If

$$\mathsf{ACA}_0 \vdash \forall X[\Theta(X) \to \exists Y \Delta(X, Y)]$$

where Θ and Δ are arithmetic, then there is an $n \in \omega$ such that

$$\mathsf{ACA}_0 \vdash \forall X[\Theta(X) \to \exists Y \in \Sigma_n^{0,X} \Delta(X, Y)].$$

Models of second-order arithmetic

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An ω-model satisfies ACA_0 iff it is a jump ideal, that is, iff it is closed under the Turing jump.

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Definition

We say that P is Weihrauch reducible to Q, P $\leq_W$ Q, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set Φ^X is an instance of Q, and for every solution $\hat{Y}$ to $\hat{X} = \Phi^X$, the set $Y = \Psi^{X \oplus \hat{Y}}$ is a solution to X.

Examples of Π^1_2-problems

We write $[X]^n$ for the collection of n-element subsets of X. A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$. A coloring of $[X]^2$ is stable if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. $H \subseteq X$ is homogeneous for c if there exists an i such that $c(s) = i$ for all $s \in [H]^n$. $L \subseteq X$ is limit-homogeneous for $c : [X]^2 \to k$ if there exists an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

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- D^2_k: every stable k-coloring of $[\mathbb{N}]^2$ has an infinite limit homogeneous set.

Reductions between example problems

Theorem (Jockusch)

Let $n \geq 2$. Then
Every computable instance of $\mathsf{RT}^n_{<\infty}$ has a Π^0_n solution.
There is a computable instance of RT^n_2 with no Σ^0_2 solution.
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Theorem (Cholak, Jockusch, and Slaman)

$\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2_{<\infty}$.

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We also have that $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$ for each k. Therefore $\mathsf{RT}^1_{<\infty}$ is true in every ω-model of RCA_0. However,

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Equivalently, $\mathsf{RT} \leq_\omega \mathsf{RT}^3_2$ but $\mathsf{RCA}_0 + \mathsf{RT}^3_2 \nvdash \mathsf{RT}$.

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- If a player is unable to make a move, their opponent wins.

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Winning G(Q $\to$ P)

Theorem (Hirschfeldt and Jockusch)

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We have that $RT^n_{<\infty} \leq_{gW} RT^n_2$ and $RT \leq_{gW} RT^3_2$.

Winning G(Q $\to$ P)

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If P $\leq_\omega$ Q then Player 2 has a winning strategy for G(Q $\to$ P). Otherwise, Player 1 has a winning strategy for G(Q $\to$ P).

Definition

We say that P is generalized Weihrauch reducible to Q and write P $\leq_{\mathrm{gW}}$ Q, if Player 2 has a computable winning strategy for G(Q $\to$ P).

We have that $\mathrm{RT}^n_{<\infty} \leq_{\mathrm{gW}} \mathrm{RT}^n_2$ and $\mathrm{RT} \leq_{\mathrm{gW}} \mathrm{RT}^3_2$.

If Player 2 has a winning strategy for G(Q $\to$ P) in at most $n+1$ many moves, then we write P $\leq^n_\omega$ Q, and likewise for gW.

Generalized Weihrauch reductions between example problems

Theorem (Hirschfeldt and Jockusch)

Let $n \geq 3$, $j \geq 1$, and m be such that $n + (j-1)(n-2) < m \leq n + j(n-2)$. Then $\mathrm{RT}^m_k \leq^{j+1}_{\mathrm{gW}} \mathrm{RT}^n_k$, but $\mathrm{RT}^m_k \not\leq^j_\omega \mathrm{RT}^n_k$. Therefore RT $\not\leq^j_\omega \mathrm{RT}^3_2$ for all j, although RT $\leq_\omega \mathrm{RT}^3_2$.

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Patey showed that for $n \geq 3$, the $\mathrm{RT}^n_k \leq^2_\omega \mathrm{RT}^n_j$ for $j < k$, but $\mathrm{RT}^n_k \not\leq^1_\omega \mathrm{RT}^n_j$. For $n = 2$, the least m such that $\mathrm{RT}^n_k \leq^m_\omega \mathrm{RT}^n_j$ approaches ∞ as k increases.

Extending Π^1_2 problems

In our definition of Π^1_2 problems, we required that instances and solutions be subsets of ω. We can extend this notion more generally as follows: Let $\mathcal{M}$ be an $\mathcal{L}_1$-structure with domain $|M|$. For $S \subseteq |M|$, we write $(\mathcal{M}, S)$ for the $\mathcal{L}_2$ structure with first order part $\mathcal{M}$ and second order part S. For an $\mathcal{L}_1$-structure $\mathcal{M}$, an $\mathcal{M}$-instance of P is an $X \subseteq |M|$ such that $(\mathcal{M}, \{X\}) \models \Theta(X)$, and a solution to X is a $Y \subseteq |M|$ such that $(\mathcal{M}, \{X, Y\}) \models \Psi(X, Y)$.

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Let $\mathcal{N}$ be a $\mathcal{L}_1$ structure. For $X_0, \ldots, X_n \in |N|$, let $\mathcal{N}[X_0, \ldots, X_n] = (\mathcal{N}, S)$, where S consists of all subsets of $|N|$ that are Δ^0_1-definable from parameters in $|N| \cup \{X_0, \ldots, X_n\}$.

The game $G^{\Gamma}(Q \to P)$

Let Γ be a set of $\mathcal{L}_2$-formulas and P and Q be Π^1_2-problems.
The Γ-reduction game $G^{\Gamma}(Q \to P)$ is defined as follows:

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Results from modified games

Proposition

Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas. Let P and Q be Π^1_2 problems. If $\Gamma \vdash Q \to P$, then Player 2 has a winning strategy for $\mathrm{G}^\Gamma(Q \to P)$. Otherwise, Player 1 has a winning strategy for $\hat{\mathrm{G}}^{\Gamma+Q}(Q \to P)$.

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For our main result, we need the mild extra assumption that Γ proves the existence of a universal Σ^0_1 formula.

The main result

Theorem

Let Γ satisfy the conditions. Let P and Q be Π^1_2-problems. If $\Gamma \vdash Q \to P$, then there is an n such that Player 2 has a winning strategy for $\hat{\mathrm{G}}^{\Gamma}(Q \to P)$ that ensures victory in at most n many moves. Otherwise, Player 1 has a winning strategy for $\hat{\mathrm{G}}^{\Gamma+Q}(Q \to P)$.

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Corollary

Let $\Gamma = \mathrm{RCA}_0 +$ all Π^1_1 formulas true over ω. If $P \not\leq^n_\omega Q$ for all n, then $\Gamma \not\vdash Q \to P$.

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Corollary

Let $\Gamma = \mathrm{RCA}_0 +$ all Π^1_1 formulas true over ω. If $P \not\leq^n_\omega Q$ for all n, then $\Gamma \nvdash Q \to P$.

Example

Let Q= RT^2_2 and P= $\mathrm{RT}^2_{<\infty}$. Patey showed that $\mathrm{RT}^2_{<\infty} \not\leq^n_\omega \mathrm{RT}^2_k$ for all n, k. Therefore $\Gamma \nvdash \mathrm{RT}^2_2 \to \mathrm{RT}^2_{<\infty}$.

Essential lemma

For $n \in \omega$, let $\Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n)$ be a formula asserting that

$$\text{if } X_0 \text{ is a P-instance then } (Y_0 = \Phi_{e_0}^{X_0} \wedge (\text{either } Y_0 \text{ is a solution to } X_0$$
$$\text{or } (Y_0 \text{ is a Q-instance and if } X_1 \text{ is a solution to } Y_0 \text{ then } (Y_1 = \Phi_{e_1}^{X_0 \oplus X_1}$$
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$$\ldots (Y_n = \Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n} \wedge Y_n \text{ is a solution to } X_0)) \cdots).$$

If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$, then there exists an $n \in \omega$ such that

$$\Gamma \vdash \forall X_0 \, \exists e_0, Y_0 \, \forall X_1 \, \exists e_1, Y_1 \cdots \forall X_n \, \exists e_n, Y_n \quad \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Extending generalized Weihrauch reducibility

Definition

We say that P is generalized Weihrauch reducible to Q over Γ and write $P \leq^{\Gamma}_{gW} Q$, if Player 2 has a computable (i.e., Δ^0_1), winning strategy for $G^{\Gamma}(Q \to P)$.

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If $P \leq^{\Gamma}_{gW} Q$, then there is an $n \in \omega$ such that Player 2 has a winning strategy for $G^{\Gamma}(Q \to P)$ that ensures victory in at most n many moves.

An example: limit-homogeneous sets

Recall that a coloring of $[X]^2$ is stable if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. $L \subseteq X$ is limit-homogeneous for $c : [X]^2 \to k$ if there exists an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

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We have the principles:
SRT^2_k: every stable k-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.
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Theorem (Chong, Lempp, and Yang)

SRT^2_2 *and* D^2_2 *are equivalent over* RCA_0.

An example: limit-homogeneous sets cont.

Theorem (Dzhafarov)

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Note that an instance of LH includes the color i to which the set is limit-homogeneous.

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Bound^* is the principle that for a simultaneous enumeration of bounded sets $F_0, \ldots, F_n$, there exists a common bound for the sets F_i.

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$\mathrm{RCA}_0 + \mathrm{LH} \equiv \mathrm{Bound}^*$.

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Theorem

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The following question remains open:
Is $\mathsf{SRT}^2_2 \leq^{\mathsf{RCA}_0}_{\mathsf{gW}} \mathsf{D}^2_2$?