Continuous Model Theory Revisited. Plenary Lecture, 2020 Annual Meeting. Association for Symbolic Logic.

H. Jerome Keisler

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Punch Line

Almost all of the model theory for metric structures carries over in a precise way to general $[0,1]$*-valued structures.*

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Variables:	$x_0, x_1, \ldots$
Connectives:	continuous functions $C\colon [0,1]^n \to [0,1]$.
Quantifiers:	sup, inf.
Terms, atomic formulas:	as in first order logic.
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General structure $\mathcal{M}$ with vocabulary V and universe M:

$F^{\mathcal{M}}\colon M^n \to M$ for each n-ary function symbol $F \in V$.
$P^{\mathcal{M}}\colon M^n \to [0,1]$ for each n-ary predicate symbol $P \in V$.
$c^{\mathcal{M}} \in M$ for each constant symbol in $c \in V$.
$\varphi^{\mathcal{M}}\colon M^{|\vec{x}|} \to [0,1]$ is defined inductively on formulas $\varphi(\vec{x})$.

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Pre-metric structure $\mathcal{M}$ with signature L:
General structure where $d^{\mathcal{M}}$ is a pseudo-metric, and
for each symbol $S \in V$, $S^{\mathcal{M}}$ is uniformly continuous with respect to $d^{\mathcal{M}}$ with the modulus given by L.
(More complicated than general structures.)
Follows that each $\varphi^{\mathcal{M}}(.)$ is uniformly continuous w.r.t. $d^{\mathcal{M}}$.

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Metric structure: Pre-metric structure where $d^{\mathcal{M}}$ is a complete metric.
Every pre-metric structure $\mathcal{M}$ has a unique completion $\overline{\mathcal{M}} \equiv \mathcal{M}$.
Canonical example: Unit ball of a Banach space.

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The following are defined as usual:
$\mathcal{M} \equiv \mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$, $Th(\mathcal{M})$.
Type of b over A: $\mathrm{tp}_{\mathcal{M}}(b/A) = Th(\mathcal{M}, \{b\} \cup A)$.
$\mathcal{M}$ is λ-**saturated** if for every $A \subseteq M$ of size $< \lambda$,
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Reduction of $\mathcal{M}$: Identify a, b if $(\mathcal{M}, a, \vec{x}) \equiv (\mathcal{M}, b, \vec{x})$ for all $\vec{x} \subseteq M$.
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Ultraproducts constructed using reduction.
Compactness Theorem proved using ultraproducts.
Monster structure: Reduced and κ-saturated of inaccessible size $\kappa > \aleph_0$.
Small means of cardinality $< \kappa$.

5. Definable Predicates

Let T be a general theory.

Definition

A sequence of formulas $\langle \varphi_k(\vec{x}) \rangle_{k \in \mathbb{N}}$ is **Cauchy** in T if

$$(\forall \varepsilon > 0)(\exists m)(\forall k \geq m) T \models \sup_{\vec{x}} |\varphi_m(\vec{x}) - \varphi_k(\vec{x})| \leq \varepsilon.$$

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If $\langle \varphi_k(\vec{x}) \rangle_{k \in \mathbb{N}}$ is Cauchy in T, then for each $\mathcal{M} \models T$, we write

$$[\lim \varphi_k]^{\mathcal{M}}(\cdot) = \lim_{k \to \infty} \varphi_k^{\mathcal{M}}(\cdot).$$

This limit always exists. $[\lim \varphi_k]^{\mathcal{M}}$ maps $M^{|\vec{x}|}$ into $[0, 1]$, and is called a **definable predicate** in $\mathcal{M}$.

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Definition

T_e is a **pre-metric expansion** of T if:

(i) T_e is a metric theory whose signature L_e is over V_D with distance D.

(ii) There is a sequence $\langle d \rangle = \langle d_k(x, y) \rangle_{k \in \mathbb{N}}$ of V-formulas Cauchy in T such that the general models of T_e are exactly the structures $\mathcal{M}_e = (\mathcal{M}, [\lim d_k]^{\mathcal{M}})$ where $\mathcal{M} \models T$.

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$\mathcal{M}_e$ is called the **pre-metric expansion** of $\mathcal{M}$ for T_e.
$\langle d \rangle$ is called an **approximate distance** for T_e. (Not unique).
Note that $D^{\mathcal{M}_e}$ is a definable predicate in $\mathcal{M}$ (defined by $\langle d \rangle$).

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These results have far-reaching consequences,
which extend most of the model theory for metric structures to general structures.

8. Absoluteness

A **property** is a class of structures closed under isomorphism.

Definition

A property $\mathcal{P}$ of general structures is **absolute** if for every general structure $\mathcal{M}$ and pre-metric expansion $\mathcal{M}_e$, $\mathcal{M}$ has property $\mathcal{P}$ if and only if $\mathcal{M}_e$ has property $\mathcal{P}$.

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If A is a set of new constant symbols, then every pre-metric expansion of T as a V-theory is also a pre-metric expansion of T as a $(V \cup A)$-theory.

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Trivial Example: For each V-formula $\varphi(\vec{x})$ and tuple $\vec{a}$ of parameters, the property $\mathcal{M} \models \varphi(\vec{a})$ is absolute.

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Being an elementary substructure is absolute. *That is, if* $\mathcal{M}, \mathcal{N} \models T$, *then* $\mathcal{M} \prec \mathcal{N}$ *iff* $\mathcal{M}_e \prec \mathcal{N}_e$.

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Being a monster structure is absolute.

10. Absolute Version of a Pre-metric Property

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Then $\mathcal{P}_1(\mathcal{M})$ iff $\mathcal{P}_1(\mathcal{M}_e)$ iff $\mathcal{Q}(\mathcal{M}_e)$ iff $\mathcal{P}_2(\mathcal{M}_e)$ iff $\mathcal{P}_2(\mathcal{M})$.

□

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D has diameter one.

$\varphi^{\mathcal{M}}$ is Lipschitz continuous with respect to D.

12. Topological and Uniform Properties

Proposition

A set $S \subseteq M^n$ being closed has an absolute version.
S is closed iff there is a set $\Phi(\vec{x})$ of V-formulas such that

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Proposition

A sequence of elements $\langle b_k \rangle$ being Cauchy has an absolute version.
$\langle b_k \rangle$ is Cauchy in $\mathcal{M}$ iff it has a limit in some $\mathcal{N} \succ \mathcal{M}$.

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Definition

A general structure $\mathcal{M}$ is a **completion** of $\mathcal{N}$ if $\mathcal{M}$ is complete and the reduction of $\mathcal{N}$ is a dense elementary substructure of $\mathcal{M}$.

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Every general structure has a unique completion up to isomorphism.

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Proposition

Every $\aleph_1$-saturated reduced general structure is complete.
So every monster structure is complete.

14. Definable and Algebraic Closure

In a metric structure, a set S is **definable over** A if S is closed and $dist(x, S) = \inf\{D(x, y) \colon y \in S\}$ is a definable predicate over A.

$b \in \mathrm{dcl}(A)$ if $\{b\}$ is definable over A.

$b \in \mathrm{acl}(A)$ if $b \in C$ for some compact C definable over A.

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$b \in \mathrm{acl}(A)$ if $b \in C$ for some compact C definable over A.

Proposition

Being a definable set over A has an absolute version.
S is definable over A iff S is closed and for each V-formula $\varphi^{\mathcal{M}}(x, y)$, if $\varphi^{\mathcal{M}}$ is a pseudo-metric then
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14. Definable and Algebraic Closure

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Proposition

$b \in \mathrm{dcl}(A)$ and $b \in \mathrm{acl}(A)$ have absolute versions.
Let $\mathcal{M}$ be reduced and $\aleph_1$-saturated.
$b \in \mathrm{dcl}(A)$ iff b is the only realization of $\mathrm{tp}(b/A)$ in $\mathcal{M}$.
$b \in \mathrm{acl}(A)$ iff the set $\{c\colon \mathrm{tp}(c/A) = \mathrm{tp}(b/A)\}$ is compact in $\mathcal{M}$.

15. Stable Theories

Definition

A complete general theory T with monster model $\mathcal{M}$ is **stable** if there is a small cardinal $\lambda < |\mathcal{M}|$ such that whenever $A \subseteq M$ and $|A| \leq \lambda$, the set of complete types over A in $\mathcal{M}$ has cardinality $\leq \lambda$.

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A **stable independence relation** is a ternary relation on small sets that satisfies Invariance, Symmetry, Transitivity, Finite Character, Full Existence, Local Character, and Stationarity.

Theorem

A complete general theory T is stable iff the monster model of T has a (unique) stable independence relation.

16. Comparing Theories via Ultrapowers

Let $\mathcal{M}, \mathcal{N}$ be general structures, T, U be complete continuous theories.

Definition

$\mathcal{D}$ **saturates** $\mathcal{M}$ if $\mathcal{D}$ is a regular ultrafilter over a set I, and the ultrapower $\mathcal{M}^I/\mathcal{D}$ is $|I|^+$-saturated.

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If $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{M} \trianglelefteq \mathcal{N}$ and $\mathcal{N} \trianglelefteq \mathcal{M}$.

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$T \trianglelefteq U$ means "$\mathcal{M} \models T \wedge \mathcal{N} \models U \Rightarrow \mathcal{M} \trianglelefteq \mathcal{N}$".
$T_{\trianglelefteq} = \{U \colon T \trianglelefteq U \text{ and } U \trianglelefteq T\}$.
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Projects: Study $(\mathbb{G}, \trianglelefteq)$. Use $\trianglelefteq$ to classify structures.

17. Known Results for first-order (FO) Theories

Let $\mathbb{F} = \{T_{\trianglelefteq} \colon T \text{ is first order}\}$. Let $T^{rg} = Th(\text{random graph})$.

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Suppose there is a super-compact cardinal.
If T is simple and $U \trianglelefteq T$, then U is simple.

18. The First Two Classes in $(\mathbb{G}, \trianglelefteq)$

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Question

Given a complete continuous theory T, is there a FO theory T_0 such that $T \trianglelefteq T_0$ and $T_0 \trianglelefteq T$? Is $h \colon (\mathbb{F}, \trianglelefteq) \to (\mathbb{G}, \trianglelefteq)$ onto?

Thank you for watching!

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