Finiteness classes inspired by Ramsey theory in choiceless set theory

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3. $X \in \mathscr{F}$ and $Y \subseteq X$ implies $Y \in \mathscr{F}$,
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All of these classes are (consistently) different from one another, as well as from the classes Fin and D-Fin.

Recall that Ramsey's theorem (which is provable in ZFC) states that, for every infinite set X, and for every colouring $c : [X]^2 \longrightarrow 2$, there exists an infinite set $Y \subseteq X$ such that $c \restriction [Y]^2$ is a constant function (we say that $[Y]^2$ is *monochromatic for* c).

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Definition

We define the class R-Fin of all sets X for which there exists a colouring $c : [X]^2 \longrightarrow 2$ such that if $Y \subseteq X$ is infinite, then $[Y]^2$ is not monochromatic for c.

Recall also that Hindman's theorem, when phrased in terms of finite unions, states that for every colouring $c : [\omega]^{<\omega} \longrightarrow 2$, there exists an infinite pairwise disjoint family $Y \subseteq [\omega]^{<\omega}$ such that the set $\mathrm{FU}(Y) = \left\{ \bigcup_{y \in F} y \middle| F \in [Y]^{<\omega} \right\}$ is monochromatic. In ZFC, we can replace $[\omega]^{<\omega}$ with $[X]^{<\omega}$ whenever X is an infinite set.

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We define the class H-Fin of all sets X for which there exists a colouring $c : [X]^{<\omega} \longrightarrow 2$ such that if $Y \subseteq X$ is infinite and pairwise disjoint, then $\mathrm{FU}(Y)$ is not monochromatic for c.

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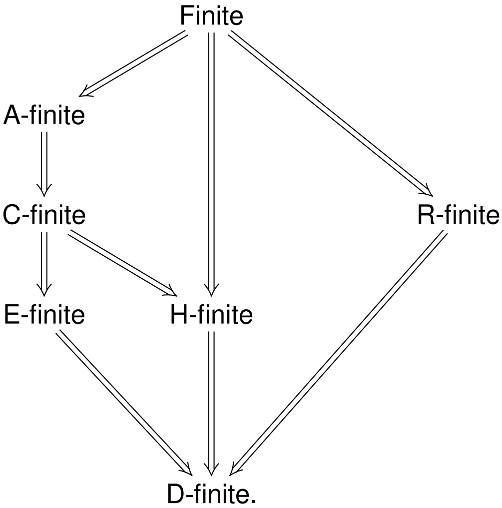

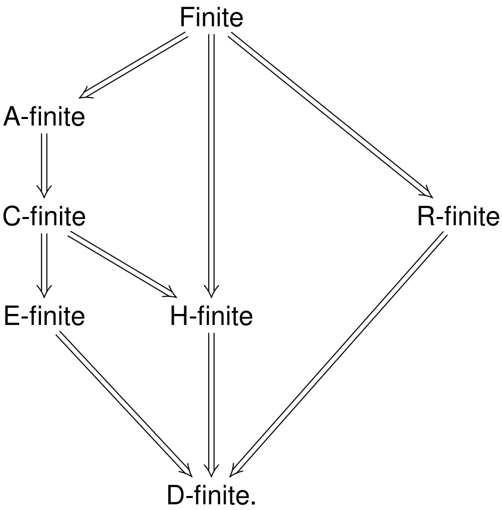

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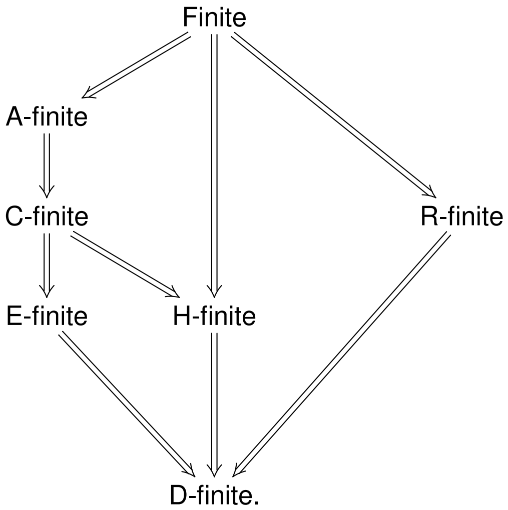

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Because ZFA includes a suitable modification of the Axiom of Foundation, in this theory we also have an analog of Zermelo's hierarchy: $V_0 = A$, $V_{\alpha+1} = \wp(V_\alpha) \cup V_\alpha$, and $V_\alpha = \bigcup_{\xi<\alpha} V_\xi$ if α is a limit ordinal.

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3. $M(A, G) \vDash$ ZFA *(but, in general, $M(A, G) \nvDash$ AC, even if we started by assuming* AC *in the real world).*

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As part of our work, we find various Fränkel–Mostowski models (some more technically complicated than others) to explicitly show that there are no further implication arrows, other than the ones in the previously shown diagram, between all of the finiteness classes under consideration.

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Thank you for reading this non-standard remote talk at this non-standard virtual conference!!!