Classes of finite structures that generate strongly minimal Steiner systems
ASL: Irvine

John T. Baldwin
University of Illinois at Chicago

March 25, 2020

Overview

1. Strongly Minimal Theories
2. Classifying strongly minimal sets and their geometries
3. Coordinatization by varieties of algebras

Thanks to Joel Berman, Gianluca Paolini, and Omer Mermelstein.

Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Klein 4-group

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Stein 4-quasigroup

Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Klein 4-group

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Stein 4-quasigroup

A Latin square is an $n \times n$ square matrix whose entries consist of n symbols such that each symbol appears exactly once in each row and each column.

By definition, this is the multiplication table of a quasigroup.

Definitions

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$.

Steiner systems are 'coordinatized' by Latin squares.

Some History

Steiner triple systems were defined for the first time by W.S.B. Woolhouse in 1844 in the Lady's and Gentlemen's Diary and he posed the question.

For which n's does an $S(2,3,n)$ exist?

Necessity: Kirkman (1847)

$n \equiv 1 or\ 3 \pmod 6$ is necessary.

Some History

Steiner triple systems were defined for the first time by W.S.B. Woolhouse in 1844 in the Lady's and Gentlemen's Diary and he posed the question.

For which n's does an $S(2,3,n)$ exist?

Necessity: Kirkman (1847)

$n \equiv 1 or\ 3 \pmod 6$ is necessary.

Sufficiency: (Bose $6n+3$, 1939) Skolem ($6n+1$, 1958)

$n \equiv 1 or\ 3 \pmod 6$ is sufficient.

Unaware of Kirkman's work, Jakob Steiner (1853) reintroduced triple systems, and as this work was more widely known, the systems were named in his honor.

Linear Spaces

Definition (1-sorted)

The vocabulary τ contains a single ternary predicate R, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems. $\boldsymbol{K}^*$ includes infinite linear spaces.

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

k-Steiner systems

1. If $|\ell| = k$ for every ℓ, we say k-Steiner system.
2. 3-Steiner = Steiner triple.

Subtle difference: regard a Steiner triple system as a *Steiner quasigroup* where $R(a, b, c)$ means $a * b = c$.

Existentially closed Steiner Systems

Barbina-Casanovas

1. The class of finite Steiner quasigroups has AP and JEP.
2. Thus, there is a Fraïssé limit M_F.
3. M_F is the prime model of T^*_{Sq}, the theory of existentially closed quasigroups.
4. T^*_{Sq} is not small.
5. $\mathrm{acl}_{T^*_{\mathrm{Sq}}} = \mathrm{dcl}_{T^*_{\mathrm{Sq}}}$.
6. T^*_{Sq} is TP_2 and $NSOP_1$

By modifying the class of structures and notion of substructure, we vary the Fraïssé construction to find k-Steiner systems and in some cases quasigroups (not necessarily Steiner) that are at the other end of the model theoretic universe.

Strongly Minimal Theories

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Classifying strongly minimal sets and their geometries

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. ‘bi-interpretable’ with an algebraically closed field (non-locally modular)

Hrushovski’s example showed there are non-locally modular which are far from being fields; the examples don’t even admit a group structure.

Classify non-locally modular geometries of SM sets

Definition: Flat geometries

A geometry given by a dimension function d is flat if the dimension of any set E covered by d-closed sets $E_1, \ldots E_n$ is bounded by applying the inclusion exclusion principle to the E_i.

Fact

If the geometry of a strongly minimal set M is flat.

1. Forking on M is not 2-ample.
2. M does not interpret an infinite group.
3. Thus, the geometry is not locally modular and so not disintegrated.

Classifying Hrushovski Construction

The acl-geometry associated with Hrushovski constructions

Work of Evans, Ferreira, Hasson, Mermelstein suggests that up to arity or more precisely, purity, (and modulo some natural conditions)

Conjectured: any two geometries associated with Hrushovski constructions are locally isomorphic.

Locally isomorphic means that after localizing one or both at a finite set, the geometries are isomorphic.
[EF11, EF12, HM18]

We are concerned not with the acl-geometry but with the Object language geometry.

'Object Language' geometries

Strong minimality asserts the 'rank' of the universe is one and imposes a combinatorial geometry whose dimension varies with the model. We study here structures which are 'geometries' in the object language. E.g.

Projective Planes: [Bal94]

There is an almost strongly minimal (rank 2) projective plane.

An example with the least possible structure in the Lenz-Barlotti class was constructed [Bal95].

In particular, the ternary function of the coordinatizing field of the example ca nnot be decomposed into an 'addition' and a 'multiplication'.

How is algebraic structure lost?

Algebraic view

1. field
2. integral domain (lose inverses)
3. matrix ring (lose commutativity)
4. alternative ring (weaken associativity)

How is algebraic structure lost?

Algebraic view

1. field
2. integral domain (lose inverses)
3. matrix ring (lose commutativity)
4. alternative ring (weaken associativity)

geometric view

1. field (Pappian plane)
2. division ring(Desarguesian plane: lose commutativity)
3. nearfield (lose left distributive)
4. quasifield(multiplication is a quasigroup with identity)
5. alternative algebra (Moufang plane: lose full associativity)
6. ternary ring (lose associativity and distributivity and even compatible binary functions, but still have inverse)

Strongly minimal linear spaces I

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality.

Strongly minimal linear spaces I

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality.

Corollary

There can be no strongly minimal affine or projective plane, since in such planes the number of lines must equal the number of planes (mod $\aleph_0$).

Strongly minimal linear spaces II

An easy compactness argument establishes

The fundamental corollary of strong minimality

If M is strongly minimal, then for every formula $\varphi(x, \overline{y})$, there is an integer $k = k_\phi$ such that for any $\overline{a} \in M$, $(\exists^{>k_\phi} x)\varphi(x, \overline{a})$ implies there are infinitely many solutions of $\varphi(x, \overline{a})$ and thus finitely many solutions of $\neg\varphi(x, \overline{a})$.

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k. So it is a K-Steiner system.
$K = \{3, 4 \ldots k\}$.

Specific Strongly minimal Steiner Systems

Definition

A *Steiner* $(v, 2, k)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(\infty, k, 2)$ Steiner-systems.
The theory is 1-ample (not locally modular) and CM-trivial (not 2-ample).

IN ENGLISH

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Hrushovski construction for linear spaces

$\mathbf{K}_0^*$ denotes the collection of finite linear systems in the vocabulary $\tau = \{R\}$.
A line in M is a maximal R-clique
$L(A)$, the lines based in A, is the collections of lines in (M, R) that contain 2 points from A.

Definition: Paolini's δ

[Pao] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

Amalgamation and Generic model

Definition

Let $A \cap B = C$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_C B$ as follows:

1. the domain of D is $A \cup B$;
2. a pair of $a \in A - C$ and $b \in B - C$ are on a line ℓ' in D if and only if there is a line $\ell \subseteq D$ based in C such that $a \in \ell$ (in A) and $b \in \ell$ (in B). Thus $\ell' = \ell$ (in D).

Definition

The countable model $M \in \hat{\boldsymbol{K}}_0$ is $(\boldsymbol{K}_0, \leq)$-*generic* if

1. If $A \leq M, A \leq B \in \boldsymbol{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$,
2. M is a union of finite closed subsets ($A_i \leq M$).

Theorem: Paolini [Pao]

There is a generic model for $\boldsymbol{K}_0$; it is ω-stable with Morley rank ω.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. $A \leq B$ if $A \subseteq B$ and there is no B_0, $A \subsetneq B_0 \subsetneq B$ with $\delta(B_0/A) < 0$.
2. C is a 0-*primitive extension* of B if $B \leq C$ and there is no $B \subsetneq C_0 \subsetneq C$ such that $B \leq C_0 \leq C$ and $\delta(C/B) = 0$.
3. We say that the 0-primitive pair C/B is *good* if for every $C' \subsetneq C$ we have that $\delta(C'/B) > 0$.
4. For any good pair (B, C), $\chi_M(B, C)$ is the number of copies of C over B appearing in M.

α is the isomorphism type of $(\{a, b\}, \{c\})$.

Overview of constructions

1. classes of finite structures
   1. $\boldsymbol{K}_0^*$: all finite linear τ-spaces.
   2. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily ≥ 0.
   3. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: μ bounds number of 'good pairs'.
2. classes of infinite structures
   1. Models of the generic model of $\boldsymbol{K}_0$ are Paolini's ω-stable linear spaces (that can be naturally thought of as rank 3 matroids)
   2. $\boldsymbol{K}_{\mu,d} = \mathrm{mod}(T_\mu)$ strongly minimal.

Introducing μ

$\boldsymbol{\alpha}$ is the isomorphism type of the good pair $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Context

Let $\mathbf{U} = \mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:

1. (i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;
2. (ii) a number $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$

The length of a line in T_μ is $\mu(\boldsymbol{\alpha}) + 2$.

T_μ is the theory of a strongly minimal Steiner $(\mu(\boldsymbol{\alpha}) + 2)$-system

If $\mu(\boldsymbol{\alpha}) = 1$, T_μ is the theory of a Steiner triple system bi-interpretable with a Steiner quasigroup.

$\boldsymbol{K}_\mu$

Definition

1. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).
2. X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0$.).
3. Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \boldsymbol{K}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ implies M is d-closed in N.
   Moreover, if $M \in \boldsymbol{K}_d^\mu$, and $B \leq M$, for any good pair (A, B), $\chi_M(A, B) = \mu(A, B)$.

Main existence theorem

Theorem (Baldwin-Paolini)[BP20]

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory T_μ.
If $\mu(\alpha) = k$, all lines in any model of T_μ have cardinality $k+2$.
Thus each model of T_μ is a Steiner k-system and $\mu(\alpha)$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. α).

Unary definable closure/elimination of imaginaries I

Definition

$A \subseteq M \models T$ has *essentially unary definable closure* if $\mathrm{dcl}(A) = \bigcup_{a \in A} \mathrm{dcl}(a)$.

Unary definable closure is a strong way to show that a theory cannot have have elimination of imaginaries. (There is no finite coding of an independent pair).
Moreover, such a theory does not interpret a quasigroup.

Unary definable closure/elimination of imaginaries II

Theorem (B-Verbovskiy)

Suppose T_μ has only a ternary predicate (3-hypergraph) R. If T_μ is either in

1. Hrushovski's original family of examples
2. or one of the B-Paoline Steiner systems

and also satisfies:
If $A \leq M \models T_\mu$,

1. $\mu \in \mathcal{U}$
2. If $\delta(C) = 2$, then $\mu(B, C) \geq 3$ except
3. $\mu(\boldsymbol{\alpha}) > 1$,

then

1. A has, at worst, essentially unary definable closure.
2. T_μ does not admit elimination of imaginaries.

Unary definable closure/ elimination of imaginaries III

This is still being written. (To be posted on Baldwin webpage and math arxiv.)

In the Hrushovski case there is an example of the Hrushovski constrution with $\mu(B, C) = 3$ and $|B| = 2$

that has non trivial definable closure of a two element set.

Work is in progress to show it does have not have elimination of imaginaries.

Coordinatization by varieties of algebras

Coordinatizing Steiner Systems

Definition

A collection of algebras V "weakly coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct a, b, c:

$$R(a, b, c) \text{ if and only if } a * b = c$$

Theorem

Every strongly minimal Steiner (2,3)-system given by T_μ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroup definable in the system.

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Let q be a prime power.
Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality q and an element $a \in F$, define a multiplication $*$ of F by $x * y = y + (x - y)a$. An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality k, each $A \in V$ determines a Steiner k-system.
(The 2-generated subalgebras.)
And each Steiner k-system admits such a coordinatization.

But we showed the coordinatization cannot be defined in the pure Steiner system.

Forcing a prime power

Theorem

If a $(2, q)$ Steiner systems is weakly coordinatized k must be a prime power.

Proof: As, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

Interpretability

Theorem

For every prime power q there is a strongly minimal Steiner q-system whose theory is interpretable in a strongly minimal block algebra.

Theorem

Let $q = p^n$ and let V be a specified variety of $(2, q)$-block algebras over F_q. Let τ' contain ternary relations R and F. For each $\mu \in \mathcal{U}$, there is a strongly minimal τ'-theory $T_{\mu',V}$ such that the reducts to R are strongly minimal q-Steiner systems and the reducts to F are strongly minimal block algebras in the variety V with each line being a copy of $F_2(V)$, the free V-algebra on two generators.

Interpretability: Details if appropriate

Fix a vocabulary $\hat{\tau}$ with ternary predicates F, R.

Theorem

Fix a variety V of block algebra with $F_2(V) = q$ If $\mu \in \mathcal{U}$, and the lines in T_μ have length $q = p^n$. There is a strongly minimal theory $T_{\mu',V}$ such that if $(A, F, R) \models T_{\mu',V}$ then $A \restriction R$ is a Steiner q-system and $A \restriction F$ is in V.

Proof: Do the construction for structures (A, F, R) in a vocabulary $\hat{\tau}$ with $\delta(A, F, R) = \delta(A, R)$.
Modify $\boldsymbol{K}_0$ to $\hat{\boldsymbol{K}}_0$ by including only structures such that every line has length 2 or q.
Expand each line by interpreting the relation F as the graph of $F_2(V)$.

For each $\hat{\tau}$ isomorphism type of $\mu(A, F, R)$ with reduct (A, R) that represents a good pair, let $\mu'(A, F, R) = \mu(A, R)$.

With this modification, we return to the usual proof.

What do we know about coordinatizing algebra

Fact

Steiner quasigroups are congruence permutable, regular, and uniform. The variety of Steiner quasigroups is not residually small. Finite members are directly decomposable.

We can show that models of the T_μ are not locally finite.

Question

1. Are these $\aleph_1$-categorical block algebas subdirectly irreducible or even simple? Surely they are not free?!
2. How does the variety associated with T_μ depend on μ?
   Note that we can certainly get different theories $\hat{T}_\mu$ for the same μ because we had to specify the variety of the block algebra.

References I

John T. Baldwin.
An almost strongly minimal non-Desarguesian projective plane.
Transactions of the American Mathematical Society, 342:695–711, 1994.

John T. Baldwin.
Some projective planes of Lenz Barlotti class I.
Proceedings of the A.M.S., 123:251–256, 1995.

John T. Baldwin and G. Paolini.
Strongly Minimal Steiner Systems I.
Journal of Symbolic Logic, 2020?
to appear JSL: http://homepages.math.uic.edu/~jbaldwin/pub/ArXiv_versionnov219.pdf.

W. Burnside.
Groups of Finite Order.
Cambridge, 1897.

References II

David M. Evans and Marco S. Ferreira.
The geometry of Hrushovski constructions, I: The uncollapsed case.
Ann. Pure Appl. Logic, 162(6):474–488, 2011.

David M. Evans and Marco S. Ferreira.
The geometry of Hrushovski constructions, II. the strongly minimal case.
J. Symbolic Logic, 77(1):337–349, 2012.

Trevor Evans.
Universal Algebra and Euler's Officer Problem.
The American Mathematical Monthly,, 86(6):466–473, 1976.

References III

Trevor Evans.
Finite representations of two-variable identities or why are finite fields important in combinatorics?
In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 135–141. North-Holland, Amsterdam, 1982.

Assaf Hasson and M. Mermelstein.
On the geometries of Hrushovski's constructions.
Fundamenta Mathematicae, 2018.
to appear.

Kitty Holland.
Model completeness of the new strongly minimal sets.
The Journal of Symbolic Logic, 64:946–962, 1999.

References IV

M. Mermelstein.
Geometry preserving reducts of hrushovskis non-collapsed construction.
Masters thesis, 2013.

Gianluca Paolini.
New ω-stable planes.
submitted.

D.J.S. Robinson.
A Course in the Theory of Groups.
Springer-Verlag, 1982.

S. Świerczkowski.
Algebras which are independently generated by every n elements.
Fund. Math., 49:93–104, 1960/1961.

References V

Sherman K Stein.
Homogeneous quasigroups.
Pacific Journal of Mathematics, 14:1091–1102, 1964.