This presentation has animations which automatically advance certain slides. This feature only works in **Adobe Acrobat full screen mode**, so for the best experience, please view in Adobe Acrobat and press Ctrl/Cmd+L to enter full screen mode.
Distality Rank

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2020 North American Annual Meeting of the ASL
Special Session on Model Theory
Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.
Understanding Unstable NIP Theories

- Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.

- It was motivated as an attempt to better understand unstable NIP theories by studying their stable and “purely unstable,” or *distal*, parts separately. This decomposition is particularly easy to see for algebraically closed valued fields:

  **Stable Part:** Residue field
  **Distal Part:** Value group
Distal NIP Theories

Distality quickly became interesting and useful in its own right, and much progress has been made in recent years studying distal NIP theories. Such a theory exhibits **no stable behavior** since it is dominated by its order-like component.

**Examples:**

- o-minimal theories
- $p$-adics
- certain expansions of o-minimal theories (Hieronymi, Nell 2017)
- the asymptotic couple of the field of logarithmic transseries (Gehret, Kaplan 2018)
Combinatorial Results

Many classical combinatorial results can be improved when study is restricted to objects definable in distal NIP structures.

- Cutting Lemma (Chernikov, Galvan, Starchenko 2018)
  
  “We believe that distal structures provide the most general natural setting for investigating questions in ‘generalized incidence combinatorics.’”

- \((p, q)\)-Theorem (Boxall, Kestner 2018)

- Szemerédi Regularity Lemma (Chernikov, Starchenko 2018)
  - Polynomial bound on partition size
  - Homogeneity
$m$-Distality
A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is **1-distal** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each *singleton* from $A$ inserts indiscernibly...
A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is \textbf{1-distal} iff: for all $A = (a_0, a_1, a_2, a_3)$, if each \textit{singleton} from $A$ inserts indiscernibly...
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\[ a_0 \rightarrow \mathcal{I}_0 \rightarrow a_1 \rightarrow \mathcal{I}_1 \rightarrow a_2 \rightarrow \mathcal{I}_2 \rightarrow a_3 \rightarrow \mathcal{I}_3 \rightarrow \mathcal{I}_4 \]
1-Distality in Pictures...

A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is **1-distal** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each singleton from $A$ inserts indiscernibly...

![Diagram of Dedekind partition with points $a_0, a_1, a_2, a_3$ and intervals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$.]

then **all** of $A$ inserts indiscernibly...
1-Distality in Pictures...

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\[
\begin{align*}
\mathcal{I}_0 & \quad \mathcal{I}_1 & \quad \mathcal{I}_2 & \quad \mathcal{I}_3 & \quad \mathcal{I}_4 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

then **all** of $A$ inserts indiscernibly...

\[
\begin{align*}
\mathcal{I}_0 & \quad \mathcal{I}_1 & \quad \mathcal{I}_2 & \quad \mathcal{I}_3 & \quad \mathcal{I}_4 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]
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A Dedekind partition $I = I_0 + I_1 + \cdots + I_4$ is **2-distal** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each pair from $A$ inserts indiscernibly...
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\[ a_0 \quad a_1 \quad a_2 \quad a_3 \]

\[ \mathcal{I}_0 \quad \mathcal{I}_1 \quad \mathcal{I}_2 \quad \mathcal{I}_3 \quad \mathcal{I}_4 \]
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\[ \begin{align*}
\mathcal{I}_0 & \quad \mathcal{I}_1 & \quad \mathcal{I}_2 & \quad \mathcal{I}_3 & \quad \mathcal{I}_4 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet &
\end{align*} \]
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\[
\begin{align*}
\mathcal{I}_0 & \quad \mathcal{I}_1 & \quad \mathcal{I}_2 & \quad \mathcal{I}_3 & \quad \mathcal{I}_4 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

then all of $A$ inserts indiscernibly...
2-Distality in Pictures...

A Dedekind partition \( \mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4 \) is **2-distal** iff: for all \( A = (a_0, a_1, a_2, a_3) \), if each **pair** from \( A \) inserts indiscernibly...

\[
\begin{align*}
\mathcal{I}_0 & \\
\mathcal{I}_1 & \\
\mathcal{I}_2 & \\
\mathcal{I}_3 & \\
\mathcal{I}_4 & \\
\end{align*}
\]

then all of \( A \) inserts indiscernibly...

\[
\begin{align*}
\mathcal{I}_0 & \\
\mathcal{I}_1 & \\
\mathcal{I}_2 & \\
\mathcal{I}_3 & \\
\mathcal{I}_4 & \\
\end{align*}
\]
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![Diagram showing a Dedekind partition with 2-distal property]

then all of $A$ inserts indiscernibly...
A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is **3-distal** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each **triple** from $A$ inserts indiscernibly...
A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is 3-distal iff: for all $A = (a_0, a_1, a_2, a_3)$, if each triple from $A$ inserts indiscernibly...
3-Distality in Pictures...

A Dedekind partition \( \mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4 \) is **3-distal** iff: for all \( A = (a_0, a_1, a_2, a_3) \), if each **triple** from \( A \) inserts indiscernibly...

\[
\begin{align*}
&\mathcal{I}_0 & \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 & \mathcal{I}_4 \\
&\downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{align*}
\]

\[
\begin{array}{c}
a_0 \\
\bullet
\end{array}
\quad
\begin{array}{c}
a_1 \\
\bullet
\end{array}
\quad
\begin{array}{c}
a_2 \\
\bullet
\end{array}
\quad
\begin{array}{c}
a_3
\end{array}
\]

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3-Distality in Pictures...

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\end{align*}
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\[ a_0 \quad a_1 \quad a_2 \quad a_3 \]

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![Diagram showing a Dedekind partition with 3-distal property (3)](image)

then **all** of $A$ inserts indiscernibly...

![Diagram showing a Dedekind partition with 3-distal property (4)](image)
A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is 3-distal iff: for all $A = (a_0, a_1, a_2, a_3)$, if each triple from $A$ inserts indiscernibly...

\[ a_0 \quad | \quad a_1 \quad | \quad a_2 \quad | \quad a_3 \]

\[ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \]

\[ \mathcal{I}_0 \quad | \quad \mathcal{I}_1 \quad | \quad \mathcal{I}_2 \quad | \quad \mathcal{I}_3 \quad | \quad \mathcal{I}_4 \]

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\[ a_0 \quad | \quad a_1 \quad | \quad a_2 \quad | \quad a_3 \]

\[ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \]

\[ \mathcal{I}_0 \quad | \quad \mathcal{I}_1 \quad | \quad \mathcal{I}_2 \quad | \quad \mathcal{I}_3 \quad | \quad \mathcal{I}_4 \]
Let $n > m > 0$.

**Definition**

We say a Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_n$ is \textit{$m$-distal} iff: for all sets $A = (a_0, \ldots, a_{n-1}) \subseteq U$, if $A$ does not insert indiscernibly into $\mathcal{I}$, then some $m$-element subset of $A$ does not insert indiscernibly into $\mathcal{I}$. 
$m$-Distality for EM-types

Let $n > m > 0$.

**Definition**

A complete EM-type $\Gamma$ is $(n, m)$-distal iff: every Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_n \models^{EM} \Gamma$ is $m$-distal.

**Lemma**

If $\Gamma$ is $(m + 1, m)$-distal, then $\Gamma$ is $(n, m)$-distal for all $n > m$.

**Proof:** Induction on $n$. □
Definition

A complete EM-type $\Gamma$ is $m$-distal iff: it is $(m + 1, m)$-distal.

Theorem

Suppose $T$ is NIP. A complete EM-type $\Gamma$ is $m$-distal if and only if there is an $m$-distal Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\text{EM}} \Gamma$. 
**Observation:** If a complete EM-type $\Gamma$ is $m$-distal, then it is also $n$-distal for all $n > m$.

**Definition**

The **distality rank** of a complete EM-type $\Gamma$, written $\text{DR}(\Gamma)$, is the least $m \geq 1$ such that $\Gamma$ is $m$-distal. If no such finite $m$ exists, we say the distality rank of $\Gamma$ is $\omega$. 
Let $m > 0$.

### Definition

A theory $T$, not necessarily complete, is \textit{\textbf{m-distal}} iff: for all completions of $T$ and all tuple sizes $\kappa$, every $\Gamma \in S^{EM}(\kappa \cdot \omega)$ is $m$-distal.

In the existing literature, a theory is called distal if and only if it is 1-distal.

### Definition

The \textit{\textbf{distality rank}} of a theory $T$, written $\text{DR}(T)$, is the least $m \geq 1$ such that $T$ is $m$-distal. If no such finite $m$ exists, we say the distality rank of $T$ is $\omega$. 
Proposition

Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If $T$ is an $\mathcal{L}$-theory with quantifier elimination, then $\text{DR}(T) \leq m$. 

This corollary helps us find examples by putting an upper bound on distality rank:

We cannot apply the proposition to groups...
Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If $T$ is an $\mathcal{L}$-theory with quantifier elimination, then $\text{DR}(T) \leq m$.

This corollary helps us find examples by putting an upper bound on distality rank:

- The theory of the random graph has distality rank 2.
Finding Examples...

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- The theory of the random graph has distality rank 2.
- The theory of the random 3-hypergraph has distality rank 3.
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**This generalizes, so...**
- The theory of the random $m$-(hyper)graph has distality rank $m$.  

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This corollary helps us find examples by putting an upper bound on distality rank:

- The theory of the random $m$-(hyper)graph has distality rank $m$.
- The theories of $(\mathbb{N}, \sigma, 0)$ and $(\mathbb{Z}, \sigma)$, where $\sigma : x \mapsto x + 1$, have distality rank 2.

\[
\begin{aligned}
&\bullet & \bullet \\
\text{I} & \xrightarrow{a} & \xrightarrow{\sigma(a)}
\end{aligned}
\]
Proposition

Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If $T$ is an $\mathcal{L}$-theory with quantifier elimination, then $\text{DR}(T) \leq m$.

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We can not apply the proposition to groups...
For example, if $T$ is the complete theory of a strongly minimal group, then $\text{DR}(T) = \omega$: 
For example, if \( T \) is the complete theory of a strongly minimal group, then \( \text{DR}(T) = \omega \):

Let \( Ia_0 \cdots a_{m-1} \) be an algebraically independent set.

\[
\begin{array}{c}
I \\
\hline
a_0 & a_1 & a_{m-1}
\end{array}
\]
For example, if $T$ is the complete theory of a **strongly minimal group**, then $\text{DR}(T) = \omega$:

Let $\mathcal{I}a_0 \cdots a_{m-1}$ be an algebraically independent set.

Let $a_m = a_0 + \cdots + a_{m-1}$, and let $A = (a_0, \ldots, a_m)$. 

\[
\begin{array}{cccc}
a_0 & a_1 & a_{m-1} & a_m \\
\mathcal{I} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]
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Let $a_m = a_0 + \cdots + a_{m-1}$, and let $A = (a_0, \ldots, a_m)$.

Now we can insert any $m$ elements of $A$ without breaking indiscernibility...

\[
\mathcal{I} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]
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\[\begin{array}{cccccc}
a_0 & a_1 & a_{m-1} & a_m \\
\bullet & & & & \\
I & \quad \longrightarrow & \bullet(\cdots) & \bullet(\cdots) & \bullet(\cdots)
\end{array}\]
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\[
\begin{array}{cccccc}
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\[
\begin{array}{cccccc}
a_0 & a_1 & a_{m-1} & a_m \\
I & \bullet & (---) & \bullet & (--- \cdots) & (---) \bullet (---)
\end{array}
\]
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Now we can insert any $m$ elements of $A$ without breaking indiscernibility...

\[ a_0 \quad a_1 \quad a_{m-1} \quad a_m \]

However, inserting **all** of $A$ breaks indiscernibility...
For example, if $T$ is the complete theory of a strongly minimal group, then $\text{DR}(T) = \omega$:

Let $\mathcal{I}a_0 \cdots a_{m-1}$ be an algebraically independent set.

Let $a_m = a_0 + \cdots + a_{m-1}$, and let $A = (a_0, \ldots, a_m)$.

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```
\begin{align*}
& a_0 \quad a_1 \quad a_{m-1} \quad a_m \\
\mathcal{I} & \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{align*}
```

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```
\begin{align*}
& a_0 \quad a_1 \quad a_{m-1} \quad a_m \\
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\end{align*}
```
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Now we can insert any $m$ elements of $A$ without breaking indiscernibility...

However, inserting all of $A$ breaks indiscernibility...
Base Change

Adding named parameters does not increase distality rank...

**Proposition**

*If $T$ is a complete theory and $B \subseteq U$ is a small set of parameters, then $\text{DR}(T_B) \leq \text{DR}(T)$.*
Base Change

Adding named parameters does not increase distality rank...

**Proposition**

If $T$ is a complete theory and $B \subseteq U$ is a small set of parameters, then 
$\text{DR}(T_B) \leq \text{DR}(T)$.

If $T$ is NIP, adding named parameters does not change distality rank...

**Base Change Theorem**

If $T$ is NIP and $B \subseteq U$ is a small set of parameters, then 
$\text{DR}(T_B) = \text{DR}(T)$. 
Type Determinacy

Let \( n > m > 0 \).

**Definition**

Given \( p \in S_A(x_0, \ldots, x_{n-1}) \), we say that the \( n \)-type \( p \) is \textit{\( m \)-determined} iff:

- it is completely determined by the \( m \)-types \( \{ q \in S_A(x_{i_0}, \ldots, x_{i_{m-1}}) : q \subseteq p \text{ and } i_0 < \cdots < i_{m-1} < n \} \)
- it contains.
Type Determinacy

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\[
\{ q \in S_A(x_{i_0}, \ldots, x_{i_{m-1}}) : q \subseteq p \text{ and } i_0 < \cdots < i_{m-1} < n \}
\]

it contains.

**Theorem**

\textit{If \( T \) is \( m \)-distal, then for any \( n \) global invariant types}

\[
p_0(x_0), \ldots, p_{n-1}(x_{n-1})
\]

\textit{which commute pairwise, their product} \( p_0 \otimes \cdots \otimes p_{n-1} \) \textit{is \( m \)-determined.}

\textbf{Furthermore, if \( T \) is NIP, the converse holds as well.}
Relationship between \( m \)-Distality and \( m \)-Dependence

Shelah introduced \textit{\( m \)-dependence} as a property of first-order theories (and formulae) which generalizes NIP:

- 1-dependence \( \iff \) NIP
- \( m \)-dependence \( \implies (m + 1) \)-dependence

New result courtesy of Artem Chernikov:

- \( m \)-distality \( \implies \) \( m \)-dependence

Conjecture:

- \( m \)-distal regularity improves \( m \)-dependent regularity
Thank You!

A link to the paper and a longer version of the slides can be found at my website...

homepages.math.uic.edu/~roland/