Ramsey properties on Fraïssé structures

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ASL North American Annual Meeting

UC Irvine, March 25–28, 2020

Research supported by DMS-1600781 and DMS-1901753

Subtitle: How forcing helps solve some problems in combinatorics.

Ramsey's Theorems

Finite Ramsey Theorem. Given $k, m, r \geq 1$, there is an $n \geq m$ such that given a coloring $c : [n]^k \to r$, there is an $X \subseteq n$ of size m such that c is constant on $[X]^k$.

$$(\forall k, m, r \geq 1) \ \ (\exists n \geq m) \ \ n \to (m)^k_r$$

Infinite Ramsey's Theorem. (finite dimensional) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

$$(\forall k, r \geq 1) \ \ \omega \to (\omega)^k_r$$

Graph Interpretation: k-hypergraphs.

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if for each $X \in [\omega]^\omega$, there is a $Y \in [X]^\omega$ such that either $[Y]^\omega \subseteq \mathcal{X}$ or else $[Y]^\omega \cap \mathcal{X} = \emptyset$.

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \to_* (\omega)^\omega$$

Fraïssé Classes and Their Limits

A collection $\mathcal{K}$ of finite structures forms a Fraïssé class if it satisfies the Hereditary Property, the Joint Embedding Property, and the Amalgamation Property.

The Fraïssé limit of a Fraïssé class $\mathcal{K}$, denoted Flim($\mathcal{K}$) or $\mathbb{K}$, is (up to isomorphism) the ultrahomogeneous structure with Age($\mathbb{K}$) $= \mathcal{K}$.

Examples. Finite linear orders $\mathcal{LO}$; $\mathrm{Flim}(\mathcal{LO}) = \mathbb{Q}$.

Finite graphs $\mathcal{G}$; $\mathrm{Flim}(\mathcal{G}) =$ Rado graph.

Finite Structural Ramsey Theory

For structures A, B, write $A \leq B$ iff A embeds into B.

A Fraïssé class $\mathcal{K}$ has the Ramsey property if

$$(\forall A \leq B \in \mathcal{K})\ (\forall r \geq 1)\ \ \mathrm{Flim}(\mathcal{K}) \to (B)^A_r$$

Some classes of finite structures with the Ramsey property:
Linear orders, complete graphs, Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Small Ramsey degrees: Bounds but not one color.

Example: Colorings of Subgraphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

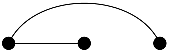


Figure: A

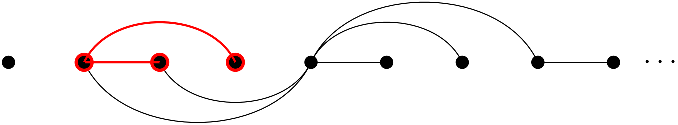
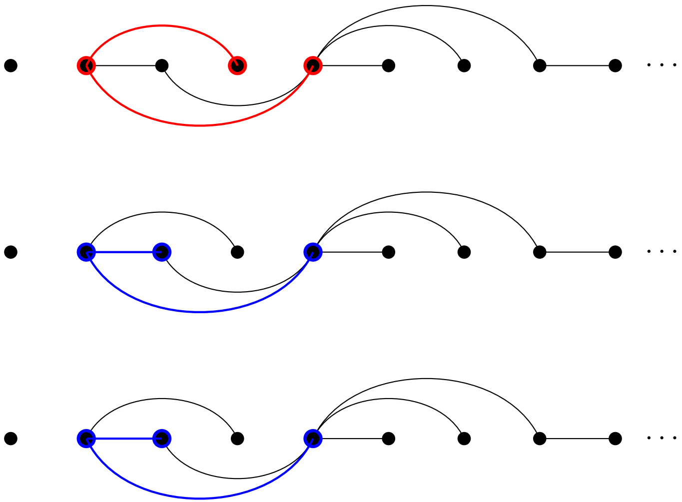


Figure: A copy of A in B

More copies of $\mathbb{A}$ into $\mathbb{B}$



Infinite Structural Ramsey Theory (finite dimensional)

Let $\mathcal{K}$ be a Fraïssé class and $\mathbb{K} = \text{Flim}(\mathcal{K})$.

(KPT 2005) For $A \in \mathcal{K}$, $T(A, \mathcal{K})$ is the least number T, if it exists, such that for each $k \geq 1$ and any coloring of the copies of A in $\mathbb{K}$, there is a substructure $\mathbb{K}' \leq \mathbb{K}$, isomorphic to $\mathbb{K}$, in which the copies of A have no more than T colors.

$$(\forall k \geq 1) \;\; \mathbb{K} \rightarrow (\mathbb{K})^A_{k, T(A,\mathcal{K})}$$

$\mathbb{K}$ has finite big Ramsey degrees if $T(A, \mathcal{K})$ is finite, for each $A \in \mathcal{K}$.

Motivation. Problem 11.2 in (KPT 2005) and (Zucker 2019).

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)
- Profinite graphs. (Huber-Geschke-Kojman, and Zheng 2018)
- Profinite k-clique-free graphs. (Dobrinen, Wang 2019)
- Structures without forbidden configurations. (Dobrinen - in progress)

Infinite Dimensional Structural Ramsey Theory

(KPT 2005) Given $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$ and some natural topology on $\mathbb{I}_{\mathbb{K}} := \binom{\mathbb{K}}{\mathbb{K}}$,

$$\mathbb{K} \to_* (\mathbb{K})^{\mathbb{K}}$$

means that all "definable" subsets of $\mathbb{I}_{\mathbb{K}}$ are Ramsey.

Motivation. Problem 11.2 in (KPT 2005).

Examples. The Baire space $[\omega]^\omega = \mathbb{I}_\omega$.

Any topological Ramsey space. But most known ones are not ultrahomogeneous structures.

(Dobrinen) The rationals, the Rado graph, and (to be checked) the Henson graphs.

Bold Conjecture

Any Fraïssé class with small Ramsey degrees has Fraïssé limit with finite big Ramsey degrees and an infinite dimensional Ramsey theorem.

Several results on big Ramsey degrees use

(1) Trees to code structures.

(2) Milliken's Ramsey theorem for strong trees, and variants.

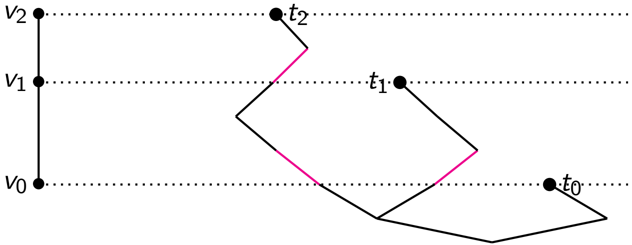
Using Trees to Code Binary Relational Structures

Rationals. $(\mathbb{Q}, <)$ can be coded by $2^{<\omega}$.

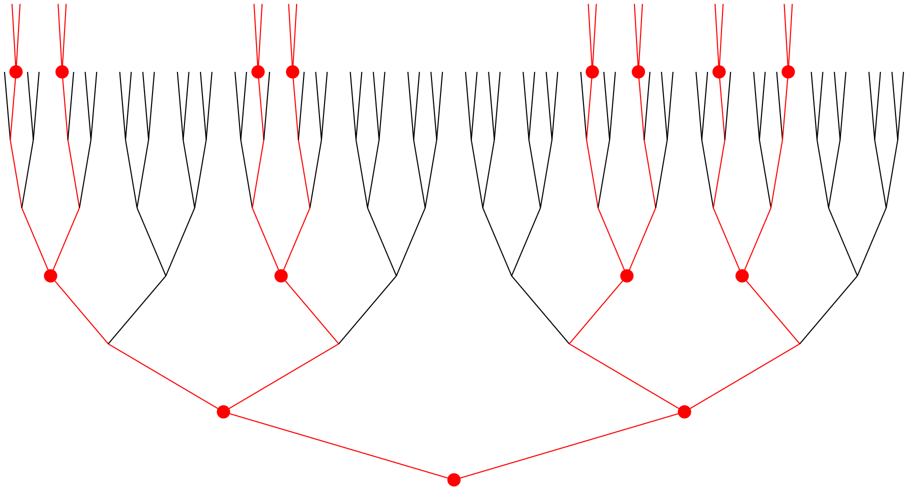
Graphs. Let A be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.

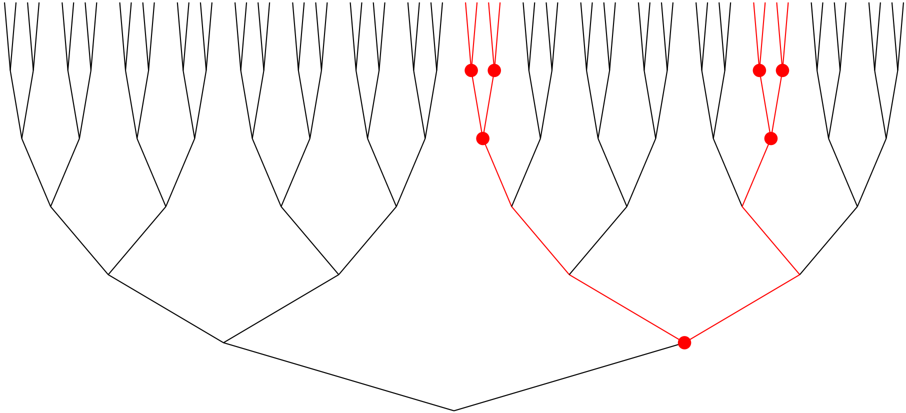


Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

A *k-strong tree* is a finite strong tree with k levels.

Thm. (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

The main tool for Milliken's theorem is the Halpern-Läuchli Theorem for colorings on products of trees.

Harrington devised a "forcing proof" of Halpern-Läuchli Theorem. This is very important to our approach to Ramsey theory on Fraïssé limits.

Halpern-Läuchli Theorem - strong tree version

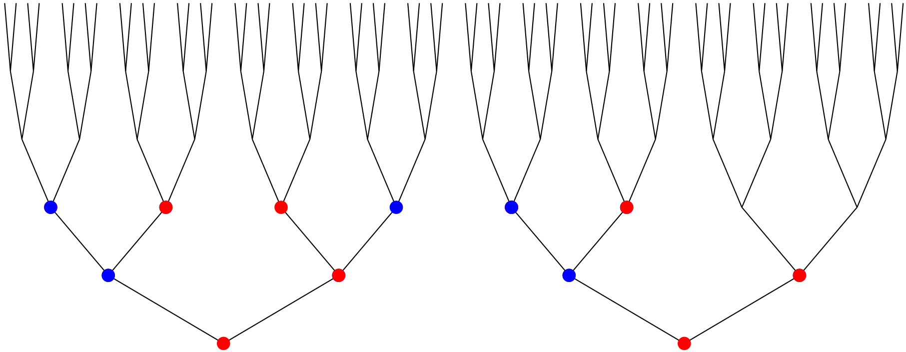
Notation: $$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

Theorem. (Halpern-Läuchli, 1966) Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes and let $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \to r$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that c is constant on $\bigotimes_{i<d} S_i$.

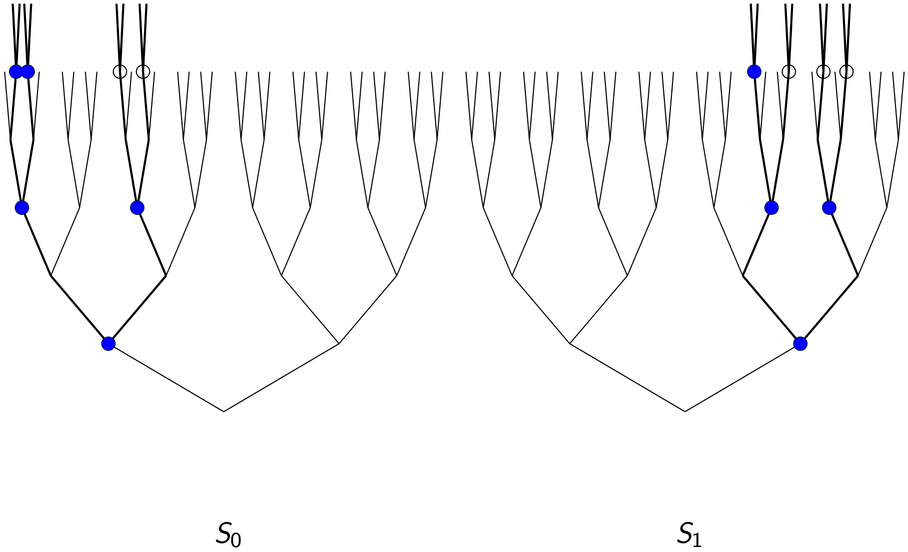
This was discovered as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971) It is also the crux of Milliken's Theorem.

We now give some examples of colorings of level products of two trees $T_0 = T_1 = 2^{<\omega}$, and show visually what the Halpern-Läuchli Theorem does.

Coloring Products of Level Sets: $T_0(0) \times T_1(0)$



HL gives Strong Subtrees with 1 color for level products



Application to Products of Rationals

Thm. (Laver, 1984) Given $d < \omega$ and a coloring of $\mathbb{Q}^d$ into finitely many colors, there are $X_i \subseteq \mathbb{Q}$, $i < d$, isomorphic to $\mathbb{Q}$ such that $X_0 \times \cdots \times X_{d-1}$ takes at most $d!$ many colors.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

Harrington devised a proof of the Halpern-Läuchli Theorem that uses forcing methods, but never goes to a generic extension.

Fix $d \geq 2$ and let $T_i = 2^{<\omega}$ $(i < d)$ be finitely branching trees with no terminal nodes. Fix a coloring $c : \bigotimes_{i<d} T_i \to 2$.

Thm. (Erdős-Rado, 1956) For $r < \omega$ and μ an infinite cardinal,

$$\beth_r(\mu)^+ \to (\mu^+)^{r+1}_\mu$$

Let $\kappa = \beth_{2d}$. Then $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$.

Harrington's 'Forcing' Proof: The Forcing

The Forcing: $\mathbb{P}$ is the set of functions p of the form

$$p : d \times \vec{\delta}_p \to \bigcup_{i<d} T_i \restriction l_p$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $l_p < \omega$, and $\forall i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

$q \leq p$ iff $l_q \geq l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $\forall (i,\delta) \in d \times \vec{\delta}_p$, $q(i,\delta) \supseteq p(i,\delta)$.

$\mathbb{P}$ adds κ branches through each tree T_i, $i < d$.

$\mathbb{P}$ is Cohen forcing adding κ new branches to each tree.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i:

$$\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P}, \text{ and } (i,\alpha) \in \mathrm{dom}(p)\}.$$

Note: If $(i,\alpha) \in \mathrm{dom}(p)$, then $p \Vdash \dot{b}_{i,\alpha} \restriction l_p = p(i,\alpha)$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on ω.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}\rangle$.

- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i,\alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, &\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\, \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf. Let $\iota \in \mathcal{I}$ be the identity function on $2d$. For any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, there are $\vec{\theta}, \vec{\theta'} \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\beta} = \iota_e(\vec{\theta'})$. Then $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta'})$ implies the conclusion. $\square$

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Pf Idea. (sliding argument) Suppose $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$.

Let $\rho_i \in \{<, =, >\}$ be the relation such that $\alpha_i \, \rho_i \, \beta_i$, $(i < d)$.

Take $\iota \in \mathcal{I}$ so that for any $\vec{\zeta} \in [K]^{2d}$ and $i < d$, $\zeta_{\iota(2i)} \ \rho_i \ \zeta_{\iota(2i+1)}$.

Fix $\vec{\theta} \in [K']^{2d}$ such that $\iota_e(\vec{\theta}) = \vec{\alpha}$ and $\iota_o(\vec{\theta}) = \vec{\beta}$.

Take $\vec{\gamma} \in [K]^d$ such that $(\forall i < d)\ \alpha_i \, \rho_i \, \gamma_i$ and $\gamma_i \, \rho_i \, \beta_i$.

Take $\vec{\mu}, \vec{\nu} \in [K]^{2d}$ with $\iota_e(\vec{\mu}) = \vec{\alpha}$, $\iota_o(\vec{\mu}) = \iota_e(\vec{\nu}) = \vec{\gamma}$, and $\iota_o(\vec{\nu}) = \vec{\beta}$.

$\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ implies $\langle j, j' \rangle$ is in the last sequence in $f(\iota, \vec{\theta})$.

$f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$ implies $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(j') = \delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(j')$,

which implies $j = j'$. $\square$

Harrington's 'Forcing' Proof: Set of compatible conditions

Main Lemma. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Pf. Suppose TAC $\exists \vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$ with $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$.
By Lem 1, for each $i < d$ and $j < k^*$, $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j))$.
So $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$ implies $\exists i < d$ and $j, j' < k^*$ with $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ but $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) \neq p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j'))$.
Note that $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j}$ and $p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j')) = t_{i,j'}$ imply $j \neq j'$.
But by Lem 2, $j \neq j'$ implies $\delta_{\vec{\alpha}}(j) \neq \delta_{\vec{\beta}}(j')$. $\rightarrow\leftarrow$ $\square$

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d \rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$.
Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

Harrington's 'Forcing' Proof: The Construction

Build strong subtrees $S_i \subseteq T_i$ homogeneous for c: Let $\mathrm{stem}(S_i) = t_i^*$.

Induction Assumption: $m \geq 1$, and we have constructed m-strong subtrees $\bigcup_{j<m} S_i(j)$ of T_i such that c takes color ε^* on $\bigcup_{j<m} \prod_{i<d} S_i(j)$.

Let X_i be the set of immediate extensions in T_i of the nodes in $S_i(m-1)$. Let $J_i \subseteq [K_i]^{|X_i|}$. Label the nodes in X_i as $\{q(i,\delta) : \delta \in J_i\}$. Let $\vec{J} = \prod_{i<d} J_i$. For each $\vec{\alpha} \in \vec{J}$ and $i < d$, $q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$. Let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair (i, γ) with $\gamma \in \vec{\delta}_q \setminus J_i$, $\exists \vec{\alpha} \in \vec{J}$ and $\exists j' < k^*$ such that $\delta_{\vec{\alpha}}(j') = \gamma$. By Main Lemma, $\vec{\beta} \in \vec{J}$ and $\gamma \in \vec{\delta}_{\vec{\beta}}$ imply that $p_{\vec{\beta}}(i,\gamma) = p_{\vec{\alpha}}(i,\gamma) = t_{i,j'}^*$. Let $q(i,\gamma)$ be the leftmost extension of $t_{i,j'}^*$ in T. This defines q. Check that $q \in \mathbb{P}$.

Note that $q \leq p_{\vec{\alpha}}$, for all $\vec{\alpha} \in \vec{J}$.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

To construct $S_i(m)$, take $r \le q$ for which $r \Vdash$ "$\forall \vec{\alpha} \in \vec{J},\ c(\dot{b}_{\vec{\alpha}} \restriction l_r) = \varepsilon^*$".

Then it is simply true in the ground model that

$$c(\{r(i, \alpha_i) : i < d\}) = \varepsilon^*, \text{ for each } \vec{\alpha} \in \vec{J}.$$

For each $i < d$, we define $S_i(m) = \{r(i, \delta) : \delta \in J_i\}$. This set extends X_i.

Then c takes value ε^* on $\prod_{i<d} S_i(m)$.

Set $S_i = \bigcup_{m<\omega} S_i(m)$. c is monochromatic on $\bigotimes_{i<d} S_i$. $\square$ HL

The Halpern-Läuchli Theorem forms the seed of the Milliken Theorem.

Big Ramsey degrees for the Rado graph and for the rationals only need Milliken's Theorem.

However, for infinite structures with some forbidden configurations, Milliken's Theorem does not suffice.

But, we can instigate a new approach using trees with coding nodes and prove new Milliken-style theorems using forcing to find bounds for big Ramsey degrees.

Theme for Ramsey Theory on Infinite Structures

1. Given an infinite relational structure $\mathbb{F}$ (a Fraïssé limit), enumerate its universe in order-type ω as $v_0, v_1, v_2, \dots$.
2. Form an associated tree $\mathbb{T}$ with special nodes which code the universe of $\mathbb{F}$; the n-th level of $\mathbb{T}$ has one node c_n which codes v_n.
3. The n-th level of the tree $\mathbb{T}$ consists of all realizable finite partial 1-types over the initial segment of the structure $\mathbb{F} \restriction \{v_i : i < n\}$.
4. Take $\mathcal{T}$ to be the collection of all subtrees of $\mathbb{T}$ which are isomorphic (in some strong sense) to $\mathbb{T}$. Each tree $T \in \mathcal{T}$ codes a copy of $\mathbb{F}$ in the same way that $\mathbb{T}$ does.
5. Develop forcing arguments to prove Ramsey theorems for antichains of coding nodes within members of $\mathcal{T}$.
6. Use these Ramsey theorems on the trees to deduce finite big Ramsey degrees for the structures, and infinite dimensional Ramsey theory.

This theme was developed to prove finite big Ramsey degrees for Henson graphs.

In hindsight it proved useful for infinite dimensional Ramsey theory on the rationals and the Rado graph, as well as the Henson graphs.

It also is proving fruitful for big Ramsey degrees on structures without forbidden configurations - a work in progress.

Henson Graphs $\mathcal{H}_k$: The k-Clique-Free Random Graph

The the k-clique-free Henson graph, $\mathcal{H}_k$, is the Fraïssé limit of the Fraïssé class of finite K_k-free graphs.

Thus, $\mathcal{H}_k$ is the ultrahomogenous K_k-free graph which is universal for all k-clique-free graphs on countably many vertices.

- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl 1986)
- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer 1989)
- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer 1998)
- For all $k \geq 3$, $\mathcal{H}_k$ has finite big Ramsey degrees. (Dobrinen 2017, 2019)

Trees with Coding Nodes

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c_n^T.

Top-down approach to Strong Coding Trees

Let $k \geq 3$ be fixed.

Order the vertices of $\mathcal{H}_k$ in order-type ω as $\langle v_n : n < \omega \rangle$.

Let the n-th coding node, c_n, code the n-th vertex.

Strong K_3-Free Tree

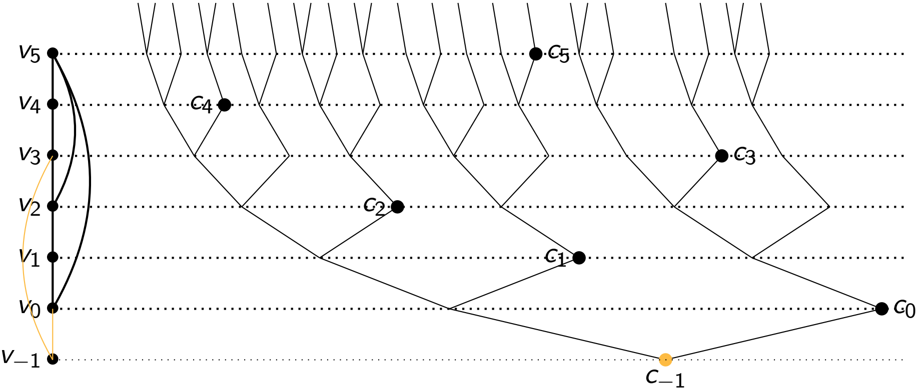


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

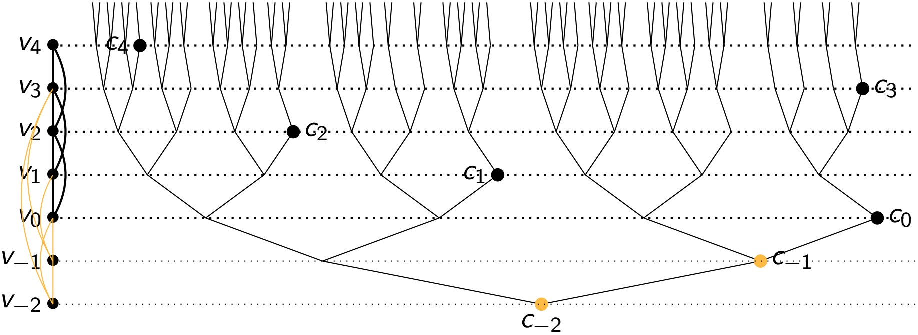


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Bottom-up Approach

Henson gave an Extension Property for building $\mathcal{H}_k$, which we can interpret in terms of trees with coding nodes, call it $(A_k)^{\texttt{tree}}$.

A tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$, $t^\frown 0 \in T$ and

(∗) $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

Thm. (D.) Suppose T is a tree with no maximal nodes satisfying the K_k-Free Branching Criterion, and the set of coding nodes dense in T. Then T satisfies $(A_k)^{\texttt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree

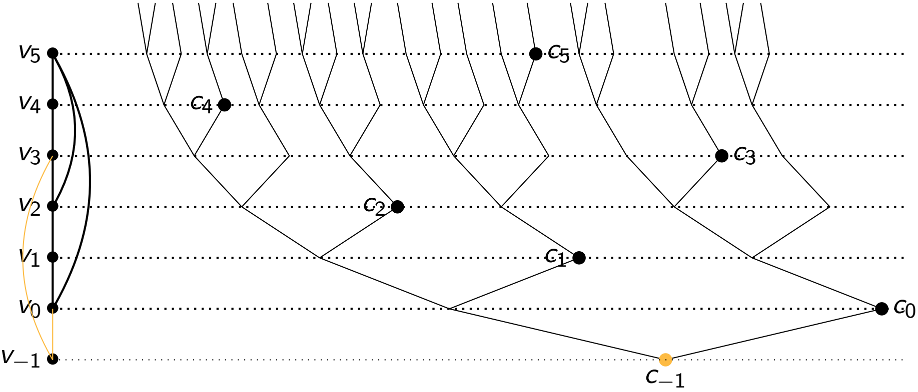


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

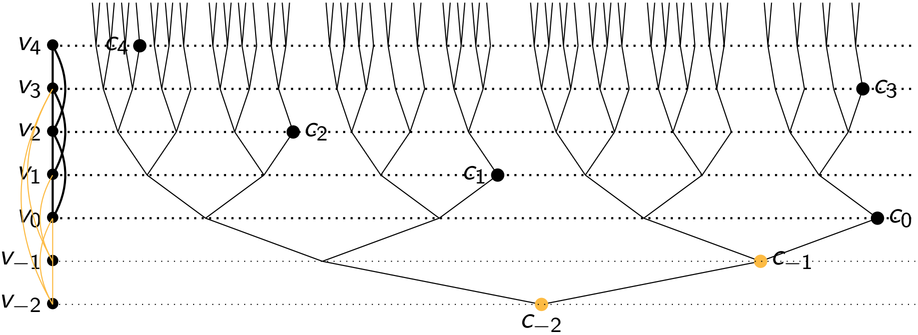


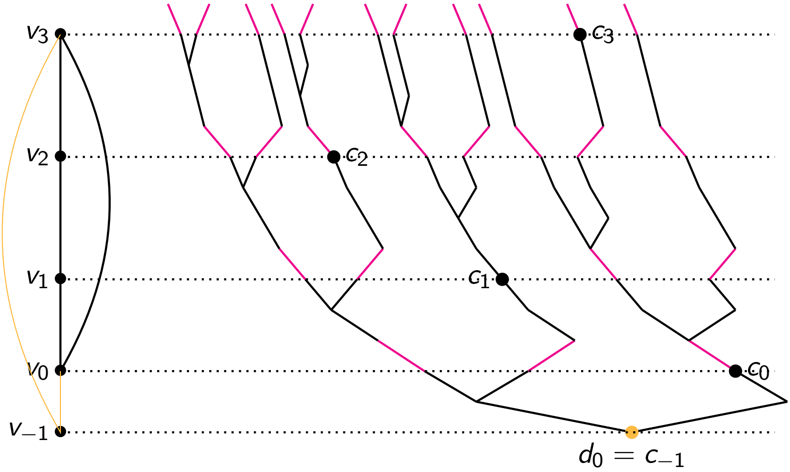
Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Almost sufficient

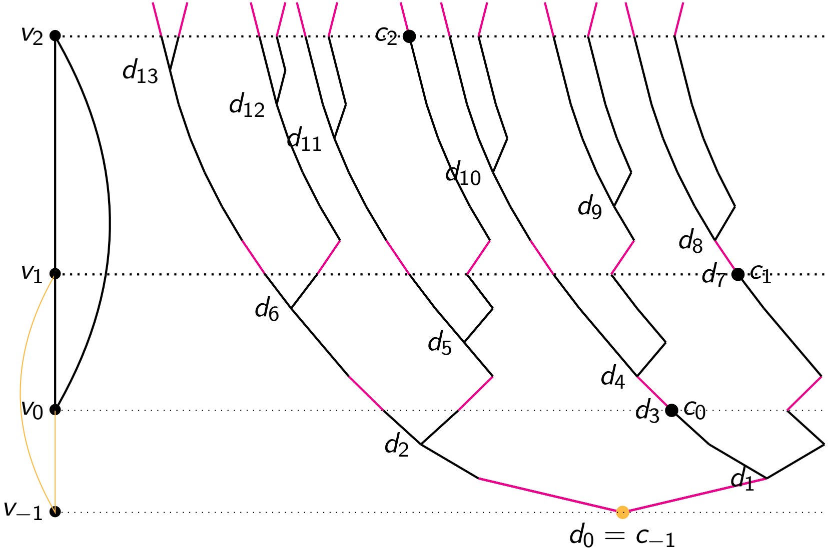
Problem: There is a bad coloring of coding nodes, which precludes indivisibility on a subcopy of $\mathcal{H}_k$ coded by any 'isomorphic' subtree coding $\mathcal{H}_k$.

Solution: Skew the levels of interest.

Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$



Defining the Space of Strong Coding Trees

Fix $k \geq 3$.

For $a \in [3, k]$, a level set $X \subseteq \mathbb{T}_k$ with nodes of length ℓ_X, has a pre-a-clique if there are $a - 2$ coding nodes in $\mathbb{T}_k$ coding an $(a - 2)$-clique, and each node in X has passing number 1 by each of these coding nodes.

Say that a subtree $T \subseteq \mathbb{T}_k$ has the Witnessing Property if each new pre-a-clique in T is witnessed by some coding node in T.

The Space of Strong $\mathcal{H}_k$-Coding Trees $\mathcal{T}_k$

Two subtrees S and T of $\mathbb{T}_k$ are strongly isomorphic iff there is a strong similarity map $f : S \to T$ which preserves maximal new pre-cliques in each interval. Such a map f is a strong isomorphism.

Idea: Strong isomorphisms preserve

1. the structure of the trees with respect to tree and lexicographic orders
2. placement of coding nodes
3. passing numbers at levels of coding nodes
4. whether or not an interval has new pre-cliques.

$\mathcal{T}_k$ = all subtrees of $\mathbb{T}_k$ which are strongly isomorphic to $\mathbb{T}_k$.

The members of $\mathcal{T}_k$ are called strong $\mathcal{H}_k$-coding trees.

Although trees with coding nodes were invented to handle forbidden cliques, it turns out they are good a coding relational structures with or without forbidden substructures.

Infinite Dimensional Ramsey Theory of the Rado Graph

Say $\mathcal{X} \subseteq [\omega]^\omega$ is completely Ramsey (CR) if for each nonempty $[s, A]$, there is a $B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$.

Thm. (Galvin-Prikry 1973) Every Borel subset of the Baire space is completely Ramsey.

Thm. (Ellentuck 1974) Each set with the property of Baire in the Ellentuck topology is completely Ramsey

Question. (KPT 2005) Which Fraïssé structures have infinite dimensional Ramsey theory for definable subsets?

Infinite Dimensional Ramsey Theory for $\mathbb{Q}$

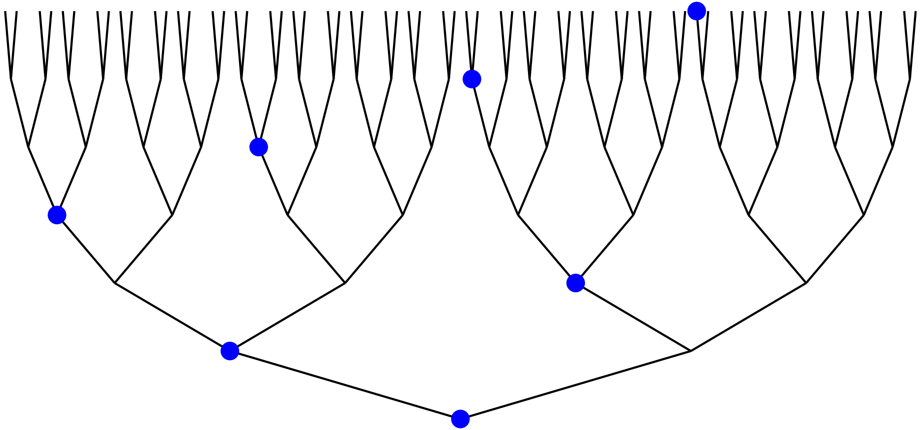
We approach this using trees with coding nodes.

By Devlin's theorem, one must fix a strong similarity type coding the rationals into $2^{<\omega}$, and restrict to all subtrees with the same strong similarity type.

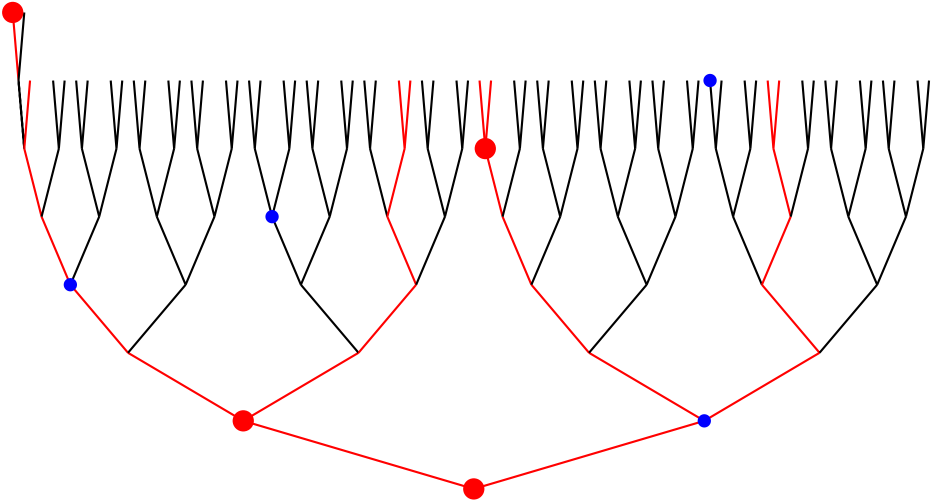
Thm. (D.) Let $\mathbb{T}_{\mathbb{Q}} \subseteq 2^{<\omega}$ be a fixed tree with coding nodes coding a copy of the rationals in order type ω, with no terminal nodes. Let $\mathcal{T}_{\mathbb{Q}}$ be the collection of all strongly similar subtrees of $\mathbb{T}_{\mathbb{Q}}$. Then $\mathcal{T}_{\mathbb{Q}}$ is a topological Ramsey space, hence has an analogue of Ellentuck's theorem.

This should also hold (modulo checking) for antichains in $2^{<\omega}$ coding the rationals. If true, this will recover Devlin's result.

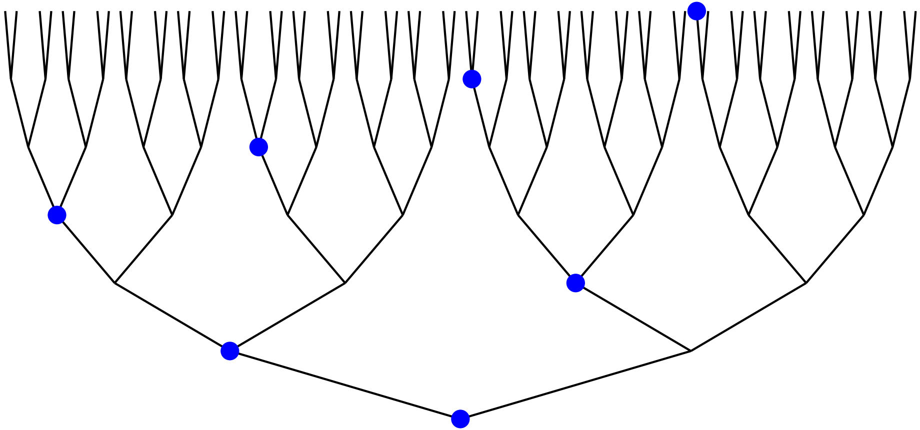
A Strong Rational Coding Tree $\mathbb{T}_{\mathbb{Q}}$



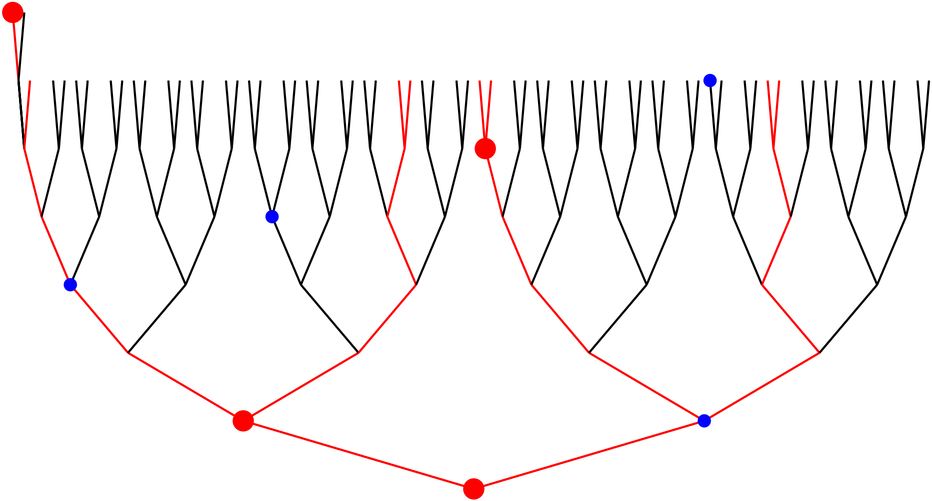
A Strong Rational Coding Subtree



A Strong Rado Coding Tree $\mathbb{T}_{\mathcal{R}}$



A Strong Rado Coding Subtree $T \in \mathcal{T}_{\mathcal{R}}$



Give $\mathcal{T}_{\mathcal{R}}$ the topology inherited as a subspace of the Cantor space.

Thm. (D.) Every Borel subset of $\mathcal{T}_{\mathcal{R}}$ has the Ramsey property.

So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory.

Fraïssé Classes Without Forbidden Configurations

It turns out that trees with coding nodes are helping determine Ramsey theory for relational structures with finitely many relations and "no forbidden configurations".

Work In Preparation: All Fraïssé classes with finitely many finitary relations satisfying a particular kind of amalgamation property have finite big Ramsey degrees.

These include structures such as hypergraphs and graphs with more than one type of edge.

Theme for Ramsey Theory on Infinite Structures

1. Given an infinite relational structure $\mathbb{F}$ (a Fraïssé limit), enumerate its universe in order-type ω as $v_0, v_1, v_2, \dots$.
2. Form an associated tree $\mathbb{T}$ with special nodes which code the universe of $\mathbb{F}$; the n-th level of $\mathbb{T}$ has one node c_n which codes v_n.
3. The n-th level of the tree $\mathbb{T}$ consists of all realizable finite partial 1-types over the initial segment of the structure $\mathbb{F} \restriction \{v_i : i < n\}$.
4. Take $\mathcal{T}$ to be the collection of all subtrees of $\mathbb{T}$ which are isomorphic (in some strong sense) to $\mathbb{T}$. Each tree $T \in \mathcal{T}$ codes a copy of $\mathbb{F}$ in the same way that $\mathbb{T}$ does.
5. Develop forcing arguments to prove Ramsey theorems for antichains of coding nodes within members of $\mathcal{T}$.
6. Use these Ramsey theorems on the trees to deduce finite big Ramsey degrees for the structures, and infinite dimensional Ramsey theory.

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