Constructive Significance of the Negative Interpretation of Classical Analysis

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An earlier version of this talk was given in June 2019 at the 12th Panhellenic Logic Symposium, Anogeia, Greece. It was dedicated to the memory of Anne S. Troelstra, 1939-2019, who contributed significantly to logic in Greece.

A preprint "Calibrating the negative interpretation," with additional proofs, will soon be posted on arXiv. This work is strongly influenced by Vafeiadou's careful analysis of weak subsystems of Kleene's formalization of intuitionistic analysis.

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Elementary Facts: In a language with all the usual logical connectives and quantifiers, Hilbert-style classical predicate logic can be formulated so that intuitionistic predicate logic has the same rules, and the same axioms except that $\neg A \rightarrow (A \rightarrow B)$ replaces the stronger, classical $\neg\neg A \rightarrow A$.

Gödel and Gentzen independently proved that classical predicate logic can be faithfully interpreted in the *negative fragment* of its intuitionistic subsystem (involving only $\&, \neg, \rightarrow$ and $\forall$), e.g. by

1. replacing predicate letters by their double negations, and
2. hereditarily replacing $A \vee B$ by $\neg(\neg A \And \neg B)$, and $\exists x A(x)$ by $\neg\forall x \neg A(x)$.

Hence: To prove that a classical theory **T** is equiconsistent with its intuitionistic subtheory **S**, it is enough to show that **S** *proves the negative interpretations of the mathematical axioms of* **T**.

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Classical arithmetic **PA** and intuitionistic arithmetic **HA**, with $=$, $0, ', +, \cdot$ and full mathematical induction, satisfy this condition.

The **Gödel-Gentzen negative interpretation** E^g of a formula E of the language of arithmetic is defined inductively:

- Prime formulas are unchanged: $(\mathrm{s} = \mathrm{t})^g \equiv (\mathrm{s} = \mathrm{t})$. (This is possible because $\vdash_{\mathbf{HA}} \neg\neg(\mathrm{s} = \mathrm{t}) \leftrightarrow (\mathrm{s} = \mathrm{t})$.)
- Negative connectives pass through: $(\forall \mathrm{x} \mathrm{A}(\mathrm{x}))^g \equiv \forall \mathrm{x} (\mathrm{A}(\mathrm{x}))^g$, $(\mathrm{A} \ \& \ \mathrm{B})^g \equiv (\mathrm{A}^g \ \& \ \mathrm{B}^g)$ and $(\mathrm{A} \rightarrow \mathrm{B})^g \equiv (\mathrm{A}^g \rightarrow \mathrm{B}^g)$.
- Disjunction $\vee$ and existence $\exists$ are interpreted classically: $(\mathrm{A} \vee \mathrm{B})^g \equiv \neg(\neg \mathrm{A}^g \& \neg \mathrm{B}^g)$ and $(\exists \mathrm{x} \mathrm{A}(\mathrm{x}))^g \equiv \neg \forall \mathrm{x} \neg (\mathrm{A}(\mathrm{x}))^g$

Theorem 1. (Gödel) **PA** and **HA** are equiconsistent.

Proof: For every arithmetical formula E:

- $\vdash_{\mathbf{PA}} (\mathrm{E} \leftrightarrow \mathrm{E}^g)$.
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Remarks:

- The negative interpretation is easily extended to a language for analysis, with variables $\alpha, \beta, \ldots$ over infinite sequences of natural numbers. Set $(\exists \alpha \mathrm{B}(\alpha))^g \equiv \neg \forall \alpha \neg (\mathrm{B}(\alpha))^g$.
- The *neutral* (classically and intuitionistically correct) basic subsystem **B** of Kleene's formal system **I** for intuitionistic analysis has mathematical axioms (countable choice and bar induction) whose negative interpretations are unprovable in **B**.
- The negative interpretation of Brouwer's continuity principle (the axiom separating **I** from **B**) is *refutable* in **B** *and* in **I**.

Question: What must be added to a subsystem **S** of Kleene's formal system **I** of intuitionistic analysis, in order to prove the negative interpretations of the classically correct axioms of **S**?

Let $\mathbf{S}^{+g}$ be the *minimum classical extension of* **S** in this sense, and let $\mathbf{S}^g$ be the *negative fragment* of $\mathbf{S}^{+g}$. (So $\mathbf{S}^{+g} = \mathbf{S} + \mathbf{S}^g$.)

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Theorem 2. If **S** $\subseteq$ **B**, then

(a) $\mathbf{S}^{+g}$ and $\mathbf{S}^{g}$ and $\mathbf{S} + (\neg\neg \mathrm{A} \to \mathrm{A})$ are equiconsistent, and have exactly the same classical ω-models as **S**.

(b) $\mathbf{S}^{+g}$ is consistent with Kleene's intuitionistic analysis **I**.

Proofs: (a): If $\mathbf{S} \subseteq \mathbf{B}$ then $\mathbf{S}^{g} \subseteq \mathbf{S}^{+g} \subseteq \mathbf{S} + (\neg\neg \mathrm{A} \to \mathrm{A})$, and for every formula E of the language of analysis:

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(b): All the axioms of **I** and all classically correct negative formulas are Kleene function-realizable. The rules of inference of **I** preserve function-realizability, and $0 = 1$ is not function-realizable.

Challenge: Clarify the classical vs. the intuitionistic mathematical content of a given subsystem **S** of Kleene's neutral analysis **B**, by finding a *nice* characterization of $\mathbf{S}^{+g}$.

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- $\mathbf{S} + (\neg\neg \mathrm{A} \to \mathrm{A}) \vdash \mathrm{E}$ if and only if $\mathbf{S}^{+g} \vdash \mathrm{E}^{g}$, which happens if and only if $\mathbf{S}^{g} \vdash \mathrm{E}^{g}$ *using only negative rules and axioms*.

(b): All the axioms of $\mathbf{I}$ and all classically correct negative formulas are Kleene function-realizable. The rules of inference of $\mathbf{I}$ preserve function-realizability, and $0 = 1$ is not function-realizable.

Challenge: Clarify the classical vs. the intuitionistic mathematical content of a given subsystem $\mathbf{S}$ of Kleene's neutral analysis $\mathbf{B}$, by finding a *nice* characterization of $\mathbf{S}^{+g}$.

Mathematical axioms of B:

- $=$ is an equivalence relation.
- 0 is not a successor, and $'$ is one-to-one.
- $\mathrm{x} = \mathrm{y} \rightarrow \alpha(\mathrm{x}) = \alpha(\mathrm{y})$.
- Primitive recursive defining equations for enough function constants to provide names for the characteristic function of Kleene's T-predicate and the result-extracting function U.
- Mathematical induction: $\mathrm{A}(0) \,\&\, \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$.
- λ-reduction: $(\lambda \mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t})$ for terms $\mathrm{r}(\mathrm{x}), \mathrm{t}$.
- Countable choice (^{x}2.1 in Kleene-Vesley 1965):

  $$\mathsf{AC}_{01} : \forall \mathrm{x} \exists \alpha \mathrm{A}(\mathrm{x}, \alpha) \rightarrow \exists \beta \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\beta(\langle \mathrm{x}, \mathrm{y} \rangle)).$$
- The "bar theorem" (^{x}26.3b in Kleene-Vesley 1965):

  $$\mathsf{BI}_1 : \forall \alpha \exists \mathrm{x} \rho(\overline{\alpha}(\mathrm{x})) = 0 \,\&\, \forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \,\&\, \rho(\mathrm{w}) = 0 \rightarrow \mathrm{A}(\mathrm{w})) \\ \&\, \forall \mathrm{w}(\mathrm{Seq}(\mathrm{w}) \,\&\, \forall \mathrm{s} \mathrm{A}(\mathrm{w} * \langle \mathrm{s}+1 \rangle) \rightarrow \mathrm{A}(\mathrm{w})) \rightarrow \mathrm{A}(\langle\, \rangle).$$

  Here $\mathrm{Seq}(\mathrm{w})$ expresses "w codes a finite sequence," $\overline{\alpha}(\mathrm{x})$ represents $\alpha(0), \ldots, \alpha(x-1)$, and $\overline{\alpha}(0) = \langle\, \rangle = 1$.

The logic of **B** *is intuitionistic.* Let $\mathbf{C} \equiv \mathbf{B} + (\neg\neg A \to A)$.

Two weak but useful subsystems of B:

Two-sorted intuitionistic arithmetic $\mathbf{IA}_1$ is the fragment of Kleene's basic system **B** obtained by omitting the axioms of countable choice and bar induction. There is full mathematical induction, but no comprehension or choice. The primitive recursive functions form a classical ω-model of $\mathbf{IA}_1$. It is easy to show $(\mathbf{IA}_1)^{+g} = \mathbf{IA}_1$.

Intuitionistic recursive analysis **IRA** adds to $\mathbf{IA}_1$ the axiom

$$\text{qf-AC}_{00} : \forall x \exists y \rho(\langle x, y \rangle) = 0 \to \exists \alpha \forall x \rho(\langle x, \alpha(x) \rangle) = 0$$

of quantifier-free countable choice, which guarantees that the class of functions is closed under "recursive in." The general recursive functions form the smallest classical ω-model, but $\mathbf{IRA}^{+g} \neq \mathbf{IRA}$.

Troelstra's **EL** (1973; Troelstra and van Dalen 1988) uses a constant rec to treat primitive recursive functionals uniformly. Otherwise it is similar to **IRA**, with full induction and qf-AC_{00}. Vafeiadou (2012) gives the precise comparison, and many others.

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Axioms stronger than qf-AC_{00} but weaker than AC_{01}:

Countable comprehension ("unique choice") is

$$\mathrm{AC}_{00}! : \forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)),$$

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Weaker than the bar theorem is the binary fan theorem:

FT_1. $\forall \alpha_{B(\alpha)} \exists x \, \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists n \forall \alpha_{B(\alpha)} \exists x \leq n \, \rho(\overline{\alpha}(x)) = 0$ (where $B(\alpha) \equiv \forall y \, \alpha(y) \leq 1$). The arithmetical functions form a classical ω-model of **IRA** + FT_1 but not of **B**. **IRA** + FT_1 proves "pointwise continuous functions on [0,1] are uniformly continuous".

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The form MP_1: $\forall\alpha(\neg\neg\exists x\,\alpha(x) = 0 \rightarrow \exists x\,\alpha(x) = 0)$ of *Markov's Principle* was rejected by Brouwer but is consistent with **I** (Kleene).

Consequences of MP_1 consistent with $\mathbf{I} + \neg\,\mathsf{MP}_1$ include the **double negation shift principles**

$$\mathsf{DNS}_1.\ \forall\rho[\forall\alpha\neg\neg\exists x\rho(\overline{\alpha}(x)) = 0 \rightarrow \neg\neg\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0],$$

$$\Sigma^0_1\text{-}\mathsf{DNS}_0.\ \forall\alpha[\forall x\neg\neg\exists y\alpha(\langle x, y\rangle) = 0 \rightarrow \neg\neg\forall x\exists y\alpha(\langle x, y\rangle) = 0],$$

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Theorem 3. (**S** with at most qf-AC_{00}, but perhaps FT_1 or BI_1)

(a) $\mathbf{IRA}^{+g} = \mathbf{IRA} + \Sigma^0_1\text{-}\mathsf{DNS}_0$.

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The form MP_1: $\forall\alpha(\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0)$ of *Markov's Principle* was rejected by Brouwer but is consistent with **I** (Kleene). Consequences of MP_1 consistent with $\mathbf{I} + \neg\, \mathsf{MP}_1$ include the **double negation shift principles**

$$\mathsf{DNS}_1.\ \forall\rho[\forall\alpha\neg\neg\exists\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \neg\neg\forall\alpha\exists\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0],$$

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and the *Gödel-Dyson-Kreisel Principle*, which is equivalent over **IRA** to the weak completeness of intuitionistic predicate logic:

$$\mathsf{GDK}.\ \forall\rho[\forall\alpha_{\mathrm{B}(\alpha)}\neg\neg\exists\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \neg\neg\forall\alpha_{\mathrm{B}(\alpha)}\exists\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0].$$

Lemma (Scedrov-Vesley) **IRA** + $\mathsf{DNS}_1 \vdash \Sigma^0_1$-$\mathsf{DNS}_0$ & GDK.

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Theorem 4. (Vafeiadou) $AC_{00}!$ is equivalent over **IRA** to the *characteristic function principle for decidable* $A(x)$:

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Weak characteristic function principles, of the form

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assert only that it is *consistent* to assume that $A(x)$ has a characteristic function. Three useful special cases are

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Theorem 6. (Solovay) Let $\mathbf{S} = \mathbf{IRA} + \mathrm{BI}_1$. Then

(a) $\mathbf{S} + \mathrm{MP}_1 \vdash \Sigma^0_1\text{-WCF}_0$, hence

(b) $\mathbf{S} + \mathrm{MP}_1 \vdash \mathrm{WCF}^{Ar}_0$, hence

(c) the classical system $\mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00} + \mathrm{BI}_1 + (\neg\neg A \to A)$ can be negatively interpreted in $\mathbf{S} + \mathrm{MP}_1$.

Robert Solovay proved this theorem in 2002, as part of his plan to show finitistically that a classical system **BI** with bar induction and arithmetical countable choice (Simpson's $\Pi^1_\infty\text{-TI}_0$) and Kleene's intuitionistic system **I** have the same consistency strength.

Kleene (1969) proved that **I** is consistent relative to **B**, and Troelstra (1973) concluded that **B** could be replaced by **S** in Kleene's statement. $\mathbf{S} + \mathrm{MP}_1$ is Π^0_2-conservative over **S** (e.g. JRM 2019) so in this sense **I** and **S** have the same consistency strength.

Analysis of Solovay's clever proof of (a) shows that MP_1 can be replaced in (c) by DNS_1, since for the negative interpretation of AC^{Ar}_{00} one needs only $\Pi^0_1\text{-WCF}_0$ rather than $\Sigma^0_1\text{-WCF}_0$.

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Solovay's original aim was to negatively interpret a subsystem **BI** of classical analysis in **B**. He described **BI** informally as follows:

"The logic is classical. There are two sorts of variables: lower case letters stand for the number variables, upper case letters for the set variables.

"There is a binary predicate $=$ in two flavors: for equality of numbers or equality of sets; there is a binary epsilon relation; there are the usual function symbols from Peano: 0, S, $+$, $\cdot$.

"There are the usual Peano axioms. Induction is in the strong form. [Arbitrary formulas of the language are allowed.] One has extensionality for sets.

"One has arithmetic comprehension. The set of n such that $\Phi(\mathrm{n})$ exists [for each Φ without bound set variables].

"The key axiom asserts that if R is a binary relation which is a linear ordering and has the property that for every non-empty subset of its field there is an R-least element, then one has *full* R-induction. ..."

Now there is a significant difference in intuitionistic mathematics between sets and sequences of natural numbers. If a set is not *detachable* (i.e. if its membership relation does not satisfy the law of excluded middle) it will not have a characteristic function.

So Solovay defined a classical variant **BI**- of **BI**, with variables over numbers and number-theoretic functions, with "the same theorems as **BI** and this is finitistically provable." **BI**- extends Peano arithmetic to the two-sorted language and replaces arithmetical comprehension and "Bar-Induction" in **BI** by "suitable variants:"

"We require that the type 1 functions contain all primitive recursive functions and that if α and β are type 1 functions and [if] γ is primitive recursive in α and β then γ is a type 1 function. [Of course, I'm being sloppy here and implicitly describing axioms by describing what the intended good models of the theory are.]

"Axiom x26.3b of Kleene's **FIM**. [Caution: for the current classical context, it makes quite a difference which version of 26.3 one takes.]" (Our BI_1 is also Kleene's x26.3b, for the same reason.)

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By "Kleene's **FIM**" he means Kleene's **I**, so **BI-** is a subsystem of a classical version of **IRA** + BI_1. In fact, **BI-** is the classical version of the closure of $\mathbf{IA}_1$ + BI_1 under "primitive recursive in."

I asked Solovay for a proof in **IRA** + BI_1 + MP_1 of the negative interpretation $(\mathrm{AC}_{00}^{Ar})^g$ of AC_{00}^{Ar}. He answered:

"I haven't tried for a direct proof. But perhaps the place to start is analysing the proof of arithmetic comprehension in **BI-**. Here is a sketch of my argument:

Let $\alpha : \omega \to \omega$. We aim to prove the existence of a β with the following properties:

1) $\beta(2\mathrm{n}) = \alpha(\mathrm{n})$;

2) $\beta(2\mathrm{n}+1) > 0$ iff $\exists \mathrm{y} \mathrm{T}^\alpha(\mathrm{n}, \mathrm{n}, \mathrm{y})$.

3) If $\beta(2\mathrm{n}+1) > 0$ then it equals $\mathrm{y}+1$ where y is least such that $\mathrm{T}^\alpha(\mathrm{n}, \mathrm{n}, \mathrm{y})$.

We first define the following ρ which will be uniformly primitive recursive in α:

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Let $\alpha : \omega \to \omega$. We aim to prove the existence of a β with the following properties:

1) $\beta(2n) = \alpha(n)$;

2) $\beta(2n + 1) > 0$ iff $\exists y T^\alpha(n, n, y)$.

3) If $\beta(2n + 1) > 0$ then it equals $y + 1$ where y is least such that $T^\alpha(n, n, y)$.

We first define the following ρ which will be uniformly primitive recursive in α:

1) If s is not a sequence number then $\rho(\mathrm{s}) = 1$.
2) Now let s be a sequence number. If for some $\mathrm{j} < \mathrm{lh}(\mathrm{s})$, we have $\mathrm{j} = 2\mathrm{k}$ and $(\mathrm{s})_\mathrm{j} \neq \alpha(\mathrm{k})$, then $\rho(\mathrm{s}) = 0$;
OR 3) if for some $\mathrm{j} < \mathrm{lh}(\mathrm{s})$ we have $\mathrm{j} = 2\mathrm{k} + 1$ and $(\mathrm{s})_\mathrm{j} = 0$ and $\exists \mathrm{y} \leq \mathrm{lh}(\mathrm{s}) \mathrm{T}^\alpha(\mathrm{k}, \mathrm{k}, \mathrm{y})$ then $\rho(\mathrm{s}) = 0$:
OR 4) if for some $\mathrm{j} < \mathrm{lh}(\mathrm{s})$, we have $\mathrm{j} = 2\mathrm{k} + 1, (\mathrm{s})_\mathrm{j} = \mathrm{m} + 1$ and m is not the least y such that $\mathrm{T}^\alpha(\mathrm{k}, \mathrm{k}, \mathrm{y})$ then $\rho(\mathrm{s}) = 0$.
OTHERWISE $\rho(\mathrm{s}) = 1$.
Now I describe the predicate $\mathrm{A}(\mathrm{x})$. [For use in [Axiom x26.3b].]
not $\mathrm{A}(\mathrm{s})$ iff
1) s is a sequence number;
2)let $\mathrm{j} = 2\mathrm{k} < \mathrm{lh}(\mathrm{s})$. Then $(\mathrm{s})_\mathrm{j} = \alpha(\mathrm{k})$.
3) let $\mathrm{j} = 2\mathrm{k} + 1,\ \mathrm{j} < \mathrm{lh}(\mathrm{s})$. Then $(\mathrm{s})_\mathrm{j} > 0$ iff $\exists \mathrm{y} \mathrm{T}^\alpha(\mathrm{k}, \mathrm{k}, \mathrm{y})$. If so letting y_k be the least such y we have $(\mathrm{s})_\mathrm{j} = \mathrm{y}_\mathrm{k} + 1$.
From the fact that not $\mathrm{A}(1)$ we conclude by bar induction that $\exists \gamma \forall \mathrm{n} \rho(\overline{\gamma}(\mathrm{n})) > 0$. [Recall that we are reasoning in the "classical" system BI-.] But then it is easy to see that this γ is our desired β."

Here Solovay uses the *classical contrapositive* of BI_1 to derive the existential conclusion $\exists\gamma\forall \mathrm{n}\rho(\overline{\gamma}(\mathrm{n})) > 0$, which allows him to conclude in **BI**- that the range of the type 1 variables is closed under the Turing jump. Arithmetic comprehension follows easily by formula induction, so AC_{00}^{Ar} is provable in the classical theory **BI**-.

In order to prove that a classical theory **T** is equiconsistent with its intuitionistic subtheory **S**, it is enough to show that **S** *proves the negative interpretations of the mathematical axioms of* **T**.

Now the intuitionistic subtheory **S** of **BI**-, obtained by simply replacing classical logic by intuitionistic logic, contains $\mathbf{IA}_1 + \mathrm{BI}_1$ and is contained in $\mathbf{IRA} + \mathrm{BI}_1$. **S** does not prove the negative interpretation of BI_1, but (as Solovay observed) $\mathbf{S} + \mathrm{MP}_1$ does.

In fact, $\mathbf{S} + \mathrm{MP}_1$ proves *for all* $\mathrm{A(w)}$ (not only for $\mathrm{A(w)}$ negative):

$$\forall\alpha\neg\neg\exists \mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0 \;\&\; \forall \mathrm{w}(\mathrm{Seq(w)} \;\&\; \rho(\mathrm{w}) = 0 \to \mathrm{A(w)})$$
$$\&\; \forall \mathrm{w}(\mathrm{Seq(w)} \;\&\; \forall \mathrm{sA}(\mathrm{w} * \langle \mathrm{s}+1\rangle) \to \mathrm{A(w)}) \to \neg\neg \mathrm{A}(\langle\,\rangle).$$

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Solovay's primitive recursive functional ρ is representable in $\mathbf{IA}_1$. Kleene's list of primitive recursive functional constants is meant to be expanded as needed; Vafeiadou observed that adding a constant and axioms for $\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{n})$ guarantees that the type-1 functions are closed under "primitive recursive in." And Σ^0_1-WCF_0 is not needed for the negative interpretation of AC^{Ar}_{00}; Π^0_1-WCF_0 is enough.

Recasting Solovay's proof using intuitionistic logic with $(\mathsf{BI}_1)^g$, instead of the classical contrapositive of BI_1, leads to

Corollary 7.

(a) $\mathbf{IA}_1 + (\mathsf{BI}_1)^g \vdash \Pi^0_1$-$\mathsf{WCF}_0$, hence

(b) $\mathbf{IA}_1 + \mathsf{BI}_1 + \mathsf{DNS}_1$ proves the weak characteristic function principle for all *negative* arithmetical formulas, hence

(c) $(\mathbf{IA}_1 + \mathsf{AC}^{Ar}_{00} + \mathsf{BI}_1)^g \subseteq \mathbf{IRA} + \mathsf{BI}_1 + \mathsf{DNS}_1$.

Open questions: Are there nice, precise characterizations of $(\mathbf{IA}_1 + \mathsf{BI}_1)^{+g}$, $\mathbf{B}^{+g}$ and intermediate systems with restricted countable choice? Can $\subseteq$ in Corollary 7(c) be replaced by $=$?

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Some References:

1. S. C. Kleene, *Formalized recursive functionals and formalized realizabiity*, Memoirs of the A.M.S. No. 89, 1969.
2. A. Scedrov and R. E. Vesley, "On a weakening of Markov's Principle," Arch. Math. Logik Vol 23, 153-160, 1983.
3. A. S Troelstra, "Intuitionistic formal systems." In Troelstra, ed. *Metamathematical Investigations of Intuitionistic Arithmetic and Analysis*, Lecture Notes in Math., Springer-Verlag, 1973. (cf. *Corrections to some publications*.)
4. A. S Troelstra, D. van Dalen, *Constructivism in Mathematics*, Vols I and II, North-Holland, 1988. (cf. *Corrections ...*)
5. J. R. Moschovakis, 'Markov's principle and subsystems of intuitionistic analysis," JSL, Vol. 84, 870-876, 2019.
6. G. Vafeiadou, *Formalizing Constructive Analysis: a comparison of minimal systems and a study of uniqueness principles*, PhD thesis, University of Athens, 2012.