Characterizing some classes of rings via superstability

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AEC background

An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of τ-structures and $\leq_{\mathbf{K}}$ is a partial order on K.

Key axioms

1. Tarski-Vaught axioms.
2. Löwenheim-Skolem-Tarski axiom.

Examples

$(R\text{-Mod}, \subseteq_R), (R\text{-Mod}, \leq_p), (R\text{-AbsP}, \leq_p)$ and $(R\text{-Flat}, \leq_p)$ where $\leq_p$ stands for pure submodel.

$M \leq_p N$ if and only if for every $\bar{a} \in M$ and ϕ an existentially quantified system of linear equations, if $N \vDash \phi[\bar{a}]$ then $M \vDash \phi[\bar{a}]$.

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Some properties

1. $\mathbf{K}$ has the *amalgamation property* (AP).
2. $\mathbf{K}$ has the *joint embedding property* (JEP).
3. $\mathbf{K}$ has *no maximal models* (NMM).

There is a semantic notion of type called *Galois-type*.

Stability

- $\mathbf{gS}(M)$ is the collection of Galois-types over M.
- $\mathbf{K}$ is λ-stable if and only if $|\mathbf{gS}(M)| \leq \lambda$ for every $M \in \mathbf{K}$ of size λ.

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Universal extension (Kolman-Shelah)

M is *universal over* N if and only if $\|M\| = \|N\|$, $N \leq_{\mathbf{K}} M$ and for any $N^* \in \mathbf{K}$ of size $\|M\|$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\longrightarrow} M$.

Limit model (Kolman-Shelah)

Let λ be an infinite cardinal and $\alpha < \lambda^+$ a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 := N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

Limit models in abelian groups (M.)

G is a (λ, α)-limit model in $(Ab, \subseteq)$ iff $G \cong \mathbb{Q}^{(\lambda)} \oplus (\oplus_p \mathbb{Z}(p^\infty)^{(\lambda)})$.

AEC background

Theorem (Shelah)

Let **K** be an AEC with JEP, AP and NMM. **K** is λ-stable if and only if there is a limit model of cardinality λ.

Uniqueness of limit models

K has *uniqueness of limit models of cardinality* λ if **K** has a limit model of cardinality λ and if given $M, N \in \mathbf{K}_\lambda$ limit models, M and N are isomorphic.

Superstability

K is *superstable* if and only if **K** has uniqueness of limit models in a tail of cardinals.

For T a complete first-order theory. $(Mod(T), \preceq)$ is superstable iff T is superstable as a first-order theory.

Setting and examples

Hypothesis

Let R be a ring and T be a first-order theory (not necessarily complete) extending the theory of left R-modules such that:

1. T has an infinite model.
2. T is closed under direct sums.

Let $\mathbf{K}^T = (Mod(T), \leq_p)$ and $|T| = |R| + \aleph_0$.

Examples

- $(R\text{-Mod}, \leq_p)$.
- $(\text{TF-Ab}, \leq_p)$.
- $(\chi, \leq_p)$ where χ is a definable category of modules.
- $(R\text{-AbsP}, \leq_p)$ for R left coherent.
- $(R\text{-Flat}, \leq_p)$ for R right coherent.

Basic results (joint work with Kucera)

Some basic results (Kucera-M.)

- $\mathbf{K}^T$ is an AEC with joint embedding, amalgamation and no maximal models.
- If $\lambda^{|T|} = \lambda$, then $\mathbf{K}^T$ is λ-Galois-stable.

Pure-injective

M is pure-injective if for every N with $M \leq_p N$ we have that M is a direct summand of N, i.e., there is M' such that $N = M \oplus M'$.

Some results about limit models (Kucera-M.)

- If M is a (λ, α)-limit model with $cf(\alpha) \geq |T|^+$, then M is pure-injective.
- If M is a (λ, ω)-limit model and N is a $(\lambda, |T|^+)$-limit model, then $M \cong N^{(\aleph_0)}$.

The theory $\tilde{T}$

Lemma (Kucera-M.)

If M and N are limit models in $\mathbf{K}^T$, then M and N are elementary equivalent.

Definition

For T a theory of modules, let $\tilde{M}_T$ be the $(2^{|T|}, \omega)$-limit model of $\mathbf{K}^T$ and $\tilde{T} = Th(\tilde{M}_T)$.

Every model of $\tilde{T}$ embeds purely into a model of $\mathbf{K}^T$.

Relationship between $\tilde{T}$ and $\mathbf{K}^T$ (M.)

- For $\lambda \geq |T|$. Same stability cardinals and limit models.
- For $\lambda \geq |T|^+$. Same saturated models.

Uniqueness of limit models and Σ-pure-injective modules

Σ-pure-injective

M is Σ-pure-injective if and only if $M^{(\aleph_0)}$ is pure-injective

Why are Σ-pure-injective modules important for us? (Gruson-Jensen-Zimmerman)

- M is Σ-pure-injective if and only if $Th(M)$ is totally transcendental.
- They are closed under elementary equivalence.
- They are closed under pure submodules.

Lemma (M.)

If there exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ, then every limit model is Σ-pure-injective.

Equivalences of superstability

Theorem (M.)

The following are equivalent

1. For every $\lambda \geq |T|$, $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.
2. $\mathbf{K}^T$ is superstable.
3. There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.
4. For every $\lambda \geq |T|$, $\mathbf{K}^T$ is λ-stable.

Further algebraic characterizations of superstability (M.)

1. $\mathbf{K}^T$ is superstable.
2. Every $M \in \mathbf{K}^T$ is pure-injective.
3. There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has a Σ-pure-injective universal model in $\mathbf{K}^T_\lambda$.

Some applications: Pure-semisimple rings

Pure-semisimple rings ('77): Every R-Module is pure-injective.

Characterizing pure-semisimple rings (M.)

R is left pure-semisimple if and only if $(R\text{-Mod}, \leq_p)$ is superstable.

Noetherian rings ('22): Every absolutely pure module is injective.

Characterizing noetherian rings(M.)

Let R be a left coherent ring.

$$R \text{ is left noetherian if and only if } (R\text{-AbsP}, \leq_p) \text{ is superstable.}$$

Noetherian rings

Characterizing noetherian rings (M.)

R is left noetherian if and only if $(R\text{-AbsP}, \leq_p)$ is superstable

One needs to check that everything still works, the main reason it does is because absolutely pure modules are closed under pure-injective envelopes.

Corollary (M.)

If $(R\text{-AbsP}, \leq_p)$ is superstable, then R-AbsP is first-order axiomatizable.

Another characterization of noetherian rings (M.)

R is left noetherian if and only if $(R\text{-Mod}, \subseteq_R)$ is superstable.

In abelian groups (M.)

$(Ab, \subseteq)$ is superstable, but $(Ab, \leq_p)$ is not superstable.

Perfect rings

Perfect rings ('60): Every flat module is a projective module.

Characterizing perfect rings (M.)

R is left perfect if and only if $(R\text{-Flat}, \leq_p)$ is superstable.

How does this case compare to previous cases?

- Flat modules are NOT first-order axiomatizable.
- Flat modules are NOT closed under pure-injective envelopes so we need to work with cotorsion modules.
- (Σ-) Cotorsion modules are NOT as nice a (Σ-)pure-injective modules.

Artinian rings ('27)

Characterizing artinian rings (M.)

R is left artinian if and only if $(R\text{-Flat}, \leq_p)$ and $(\text{Mod-}R, \subseteq_R)$ are superstable.

References

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Thank you!