Proof strength of Tree Theorem

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Ramsey degree

Theorem 1 (Classical Ramsey theorem)

Every k-coloring C of $[\omega]^d$ admit an infinite set G such that such that $|C([G]^d)| = 1$.

Theorem 2 (Devlin [3])

For every k-coloring C of $[\mathbb{Q}]^d$, there exists a $X \subseteq \mathbb{Q}$ with $(X, <) \cong (\mathbb{Q}, <)$ such that $|C([X]^d)| \leq r_d$. Where $r_d \in \omega$ only depends on d.

Definition 3 (Ramsey degree)

Given a structure $\mathcal{U}$, a finitely generated substructure A of $\mathcal{U}$, the *Ramsey degree* of A with respect to $\mathcal{U}$ is the smallest $r \in \omega$ such that the following is true. For every coloring $C : Embed(A, \mathcal{U}) \to k$, there exists a substructure $\mathcal{G}$ of $\mathcal{U}$ with $\mathcal{G} \cong \mathcal{U}$ such that $|C(Embed(A, \mathcal{G}))| \leq r$.

Let $\mathcal{R}$ be the universal countable homogeneous graph.

Theorem 4

- *Ramsey degree of edge (with respect to $\mathcal{R}$) is larger than* 1 *(Erdos, Hajnal and Posa[6]);*
- *Edge has Ramsey degree* 2 *with respect $\mathcal{R}$ (Pouzet and Sauer [10]);*
- *Every finite graph has finite Ramsey degree with respect to $\mathcal{R}$(Sauer [11])*

Definition of Tree Theorem

Consider the structure $(2^{<\omega}, \prec)$ where $\prec$ denote the string extension relation. By Chubb, Hirst and McNicholl[2]:

Statement 5 (Tree theorem for pairs (TT^d_k))

Let $A_d = \{\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_{d-1}\}$. For every $d \in \omega$, A_d has Ramsey degree 1 with respect to $(2^{<\omega}, \prec)$.

Clearly each of the following Ramsey type theorem has a "tree version": $\mathsf{CAC}, \mathsf{ADS}, \mathsf{EM}$ and TS^2_k. For example, the Tree version of CAC (TCAC) states that

Statement 6 (TCAC)

Every partial order $<^*$ on $2^{<\omega}$ admit an infinite perfect tree G such that either for every two comparable string $\sigma, \rho \in G$, σ, ρ are $<^*$-comparable, or for every two comparable string $\sigma, \rho \in G$, σ, ρ are not $<^*$-comparable.

Remark 7

Despite being an interesting statement itself, many results of Ramsey degree are proved by Milliken's tree theorem (another type of tree theorem), which is proved subsequently by Halpern-Lauchli theorem. Our study of Tree theorem gives insight on proof strength of Milliken's tree theorem. For example, the k-hierarchy construction we use, gives a proof of Halpern-Lauchli theorem in RCA.

Definition of Erdos-Rado Theorem

Frittaion and Patey [7] proposed to study the Erdos-Rado theorem (ER^2_2):

Statement 8 (ER^2_2 [5])

Every 2-coloring C of $[\mathbb{Q}]^2$ admit a set G such that either G is infinite and $C([G]^2) = 0$, or G is of order type $(\mathbb{Q}, \leq)$ and $C([G]^2) = 1$.

[5] observes that ER^2_2 is close to TT^2_k (probably because that a string can be seen as an interval of reals).

Known results

Dzhafarov and Hirst [4] observed that TT^2_k admit CJS-decomposition.
Let $[2^{<\omega}]^2 = \{\{\sigma, \rho\} : \sigma \text{ is comparable to } \rho\}$.

Definition 9

A coloring $C : [2^{<\omega}]^2 \to k$ is *stable* iff: for every σ, there exists a $i \in k$ such that $C(\{\sigma, \rho\}) = i$ for all but finitely many $\rho \in [\sigma]^{\preceq}$.
A coloring $C : [2^{<\omega}]^2 \to k$ is *weakly stable* iff: for every σ, there exists a $N \in \omega$ such that for every $\rho \in [\sigma]$ with $|\rho| = N$, there exists a $i \in k$ such that $C(\{\sigma, \rho'\}) = i$ for all $\rho' \in [\rho]^{\preceq}$.

Known results

Definition 10

- *Stable Tree Theorem for pairs* (STT^2_k): Every stable coloring $C : [2^{<\omega}]^2 \to k$ admit an infinite perfect tree G such that $|C([G]^2)| = 1$.
- *Cohesive Tree Theorem for pairs* (CTT^2_k): Every coloring $C : [2^{<\omega}]^2 \to k$ admit an infinite perfect tree G such that C restricted on G is stable.
- *Weak Cohesive Tree Theorem for pairs* (wCTT^2_k): Every coloring $C : [2^{<\omega}]^2 \to k$ admit an infinite perfect tree G such that C restricted on G is weakly stable.

Proposition 11 ([4])

Over RCA_0, TT^2_k *is equivalent to* $\mathsf{STT}^2_k + \mathsf{CTT}^2_k$.

Known results

Obviously, every Ramsey type theorem is implied by its tree version. Meanwhile, Dzhafarov,Frittaion and Patey [7], [5] proved that:

Theorem 12

Over RCA_0*:*

- *[7]* RT^2_k *does not imply* TT^2_k *or* ER^2_2*;*
- *[5]* TT^2_k *does not imply* ACA_0*.*

This makes TT^2_k the first natural theorem whose proof strength strictly lies between RT^2_k and ACA_0.

Known results

Question 13

- (Dzhafarov, Patey) Over RCA_0, does TT^2_k imply WKL_0?
- (Dzhafarov, Patey) Over RCA_0, does ER^2_2 imply ACA_0?
- Does every computable TT^2_k instance admit low$_2$ solution?

In a word, which result of RT^2_k holds for their tree version and which result does not. Since ER^2_2 is close to TT^2_k, it's also natural to ask:

Question 14 (Dzhafarov, Patey)

Does TT^2_k imply ER^2_2?

Theorem 15 (Chong,Li,Liu,Yang)

Over RCA_0*:*

- TT^2_k *does not imply* WKL_0*;*
- ER^2_2 *does not imply* ACA *and* STT^2_k *does not imply* ER^2_2*;*
- *Every computable* TT^2_k *instance admit low*$_2$ *solution;*
- TADS *does not imply* CAC *and* TCAC *does not imply* SRT^2_k *(an alternative approach to* CAC *vs* ADS *then that of LST);*
- *Every computable* STCAC *instance admit a low solution;*
- TTS^2_{k+1} *does not imply* TTS^2_k*;*
- *Every* TT^1_k *instance admit generalized low solution.*

We reprove the following result of Dzhafarov and Patey to demonstrate the the CJS-style Seetapun forcing for TT^1_k.

Theorem 16 ([5])

TT^1_k *admit strong cone avoidance. i.e., for every* TT^1_k *instance* C *there exists a solution* G *such that* G *does not compute* D^* *where* D^* *is a given incomputable degree.*

Forcing condition

For two trees F', F, we write $F' \succeq F$ if $F \subseteq F'$ and $F' \setminus F \subseteq [lvs(F)]^{\preceq}$ (where $lvs(F)$ is the set of leaves of F and we rules that $lvs(\emptyset) = \{\varepsilon\}$).

Definition 17

- A *condition* is a triple (F, T, D) where F is finite and perfect, T is 2-branching over each leaf of F such that $T \leq_T D$ and $D^* \not\leq_T D$.
- A condition (F, T, D) *forces* a requirement $\mathcal{R}_\Psi$ if for every $G \succeq F$, if $G \setminus F \subseteq T$, then G satisfy $\mathcal{R}_\Psi$.
- A condition (F', T', D') *extends* (F, T, D) (written as $(F', T', D') \leq (F, T, D)$) if $D' \geq_T D$, $F' \succeq F$, $T' \subseteq T$ and $F' \setminus F \subseteq T$.

As usual, a condition (F, T, D) is seen as a collection of solutions, namely $\{F' \succeq F : F' \setminus F \subseteq T, F' \text{ is 2-branching}\}$.

Forcing condition

Definition 18

- A condition (F, T, D) *forces* $\neg\mathcal{R}$ *on color* i *of* C if $F \subseteq C^{-1}(i)$ and for some Turing functional Ψ, there exists no extension of (F', T', D') of (F, T, D) with $F' \subseteq C^{-1}(i)$ that forces $\mathcal{R}_\Psi$.
- A condition $(\emptyset, T, D)$ *potentially forces* $\neg\mathcal{R}$ if there exists $(F', T', D') \leq (\emptyset, T, D)$ that forces $\neg\mathcal{R}$.

Forcing condition

Obviously:

Lemma 19

If there exists a condition $(\emptyset, T, D)$ *and a* $i \in k$ *such that* $(\emptyset, T, D)$ *does not potentially force* $\neg\mathcal{R}$ *on color* i *of* C*, then there exists an infinite perfect* $G \subseteq T \cap C^{-1}(i)$ *such that* $D^* \not\leq_T G$.

k-hierarchy

The key Lemma is the following.

Definition 20

Let $\mathcal{F}_0, \cdots, \mathcal{F}_{k-1}$ be k sets of tree. We say $\mathcal{F}_0, \cdots, \mathcal{F}_{k-1}$ is a *k-hierarchy* iff: $\mathcal{F}_0 \neq \emptyset$ and for every $i < k-1$, every $F \in \mathcal{F}_i$, every $\sigma \in lvs(F)$, there exists $F' \in \mathcal{F}_{i+1}$ such that $F' \subseteq [\sigma]^{\preceq}$.

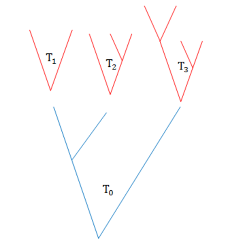

Figure: A 2-hierarchy where $\mathcal{F}_0 = \{T_0\}, \mathcal{F}_1 = \{T_1, T_2, T_3\}$.

Lemma 21

Let $\mathcal{F}_0, \cdots, \mathcal{F}_{k-1}$ be a k-hierarchy and suppose for every $i \in k$, every $F \in \mathcal{F}_i$, (F, T, D) is a condition. Then there exists an $i \in k$, a $F \in \mathcal{F}_i$ such that (F, T, D) does not force $\neg\mathcal{R}$ on color i of C.

Proof of Theorem 16

Assuming Lemma 21 and let's prove Theorem 16. If for some $i \in k$, some condition $(\emptyset, \tilde{T}, \tilde{D})$ does not potentially forces $\neg\mathcal{R}$ on color i of k, then we are done by Lemma 19. Suppose this is not the case. Then let (F_0, T_0, D_0) be a condition in $C^{-1}(0)$ that forces $\neg\mathcal{R}$. For every $\sigma \in lvs(F_0)$, let $(F_1^\sigma, T_1^\sigma, D_1^\sigma) \leq (\emptyset, T_0 \cap [\sigma]^{\preceq}, D_0)$ in $C^{-1}(1)$ that forces $\neg\mathcal{R}$. Moreover, we construct $(F_1^\sigma, T_1^\sigma, D_1^\sigma)$ in successive order so that for some degree $D_1 \not\geq_T D^*$, we have: $T_1 = \cup_{\sigma \in lvs(F_0)} T_1^\sigma \leq_T D_1$ and $D_0 \leq_T D_1$ (this is the reason we incorporate a degree in the condition). Note that $\{F_0\}, \{F_1^\sigma\}_{\sigma \in lvs(F_0)}$ is a 2-hierarchy. Keep doing this, we finally obtain a k-hierarchy and a tree T_{k-1} such that violate Lemma 21.

Proof of Lemma 21

Now we prove Lemma 21. Assume otherwise that for every $i \in k$, every $F \in \mathcal{F}_i$, (F, T, D) forces $\neg\mathcal{R}$ with Turing functional Ψ_F being a witness. Consider the question: Does every coloring $\hat{C} : 2^{<\omega} \to k$ admit two finite perfect tree $\hat{F}_0, \hat{F}_1$, some $i \in k$ and $F \in \mathcal{F}_i$ such that $\hat{F}_0, \hat{F}_1 \succeq F$, $\hat{F}_0, \hat{F}_1 \setminus F \subseteq T \cap \hat{C}^{-1}(i)$ and $\Psi_F^{\hat{F}_0}(n) \downarrow \neq \Psi_F^{\hat{F}_1}(n) \downarrow$ for some n.

If the answer is yes, substitute $\hat{C}$ by C, we have that for some $i \in k$ and $F \in \mathcal{F}_i$, $(\hat{F}, T, D)$ extends (F, T, D) and forces $\mathcal{R}_{\Psi_F}$, a contradiction.

Proof of Lemma 21

If the answer is no, let $\tilde{C}$ be a witness with $\tilde{C} \oplus D \not\geq_T D^*$; and let $\tilde{T} \leq_T \tilde{C} \oplus D$ be such that $\tilde{T} \subseteq T$ and for every $\sigma \in lvs(\cup_{i \in k, F \in \mathcal{F}_i} F)$, there exists $i_\sigma \in k$ such that $\tilde{T} \cap [\sigma]^{\preceq} \subseteq T \cap \hat{C}^{-1}(i_\sigma)$.
The key point of k-hierarchy is that there must exist a $i \in k$, $F \in \mathcal{F}_i$, $\vec{\tau} \succeq lvs(F)$ such that for every $\tau \in \vec{\tau}$, $i_\tau = i$. Thus it is easy to check that $(F, \tilde{T} \cap [\vec{\tau}]^{\preceq}, \tilde{C} \oplus D)$ is an extension of (F, T, D) that forces $\mathcal{R}_{\Psi_F}$.

Applications: Preserving hyperimmune

Applying the CJS-style Seetapun forcing for TT^1_k, we directly have the following generalization of [9]

Theorem 22

Given $k-1$ many hyperimmune set $X_j, j \leq k-1$, any TTS^1_k instance $C : 2^{<\omega} \to k$, there exists a solution G of C such that $X_j, j \leq k-1$ are all G-hyperimmune.

Applications: Preserving hyperimmune

Corollary 23

Over RCA_0, TTS^2_{k+1} *does not imply* TTS^2_k.

Proof.

Let $\tilde{C} : 2^{<\omega} \to k$ be a Δ^0_2 coloring induced by some stable TTS^2_k instance such that $\tilde{C}^{-1}(i)$ is hyperimmune for all $i \in k$. By Theorem 22, and using the CJS decomposition of TTS^2_{k+1} there is a model of TTS^2_{k+1} such that relativize to every member of that model, $\tilde{C}^{-1}(i)$ is hyperimmune for all $i \in k$. But any model of TTS^2_k contains a member G such that $\tilde{C}^{-1}(i)$ is not G-hyperimmune for some $i \in k$. □

Applications: Genearlized low solution

The concept "force $\neg\mathcal{R}$" is non constructive. To construct a generalized solution of a given TT^1_2 instance C, we carry out a constructive version the mentioned CJS-style Seetapun forcing for TT^1_2. A single step of this construction is the following.

Let $F \subseteq C^{-1}(0)$ be a finite perfect tree and for each $\sigma \in lvs(F)$, let $F^\sigma \subseteq [\sigma]^{\preceq} \cap C^{-1}(1)$ be a finite perfect tree. Let $\varphi_F, \varphi_{F^\sigma}$ be Turing functional on specific input.

We ask: Does there exists a $\hat{C} : 2^{<\omega} \to 2$ such that for every finite perfect tree $\hat{F}$, if $\hat{F} \succeq F$ and $\hat{F} \setminus F \subseteq \hat{C}^{-1}(0)$, then $\varphi_F^{\hat{F}} \uparrow$; and if for some $\sigma \in lvs(F)$, $\hat{F} \succeq F^\sigma \wedge \hat{F} \setminus F^\sigma \subseteq [\sigma]^{\preceq} \cap \hat{C}^{-1}(1)$, then $\varphi_{F^\sigma}^{\hat{F}} \uparrow$.

Applications: Genearlized low solution

If the answer is yes, substitute $\hat{C}$ by C, either we make a progress on color 0, i.e., $\hat{F} \succeq F \wedge \hat{F} \setminus F \subseteq C^{-1}(0) \wedge \varphi_F^{\hat{F}} \downarrow$; or for some $\hat{\sigma} \in lvs(F)$, $\hat{F} \succeq F^{\hat{\sigma}} \wedge \hat{F} \setminus F^{\hat{\sigma}} \subseteq [\hat{\sigma}]^{\preceq} \cap C^{-1}(1) \wedge \varphi_{F^{\hat{\sigma}}}^{\hat{F}} \downarrow$ (progress on color 1).

If the answer is no, let $\hat{C}$ be a witness such that $\hat{C}$ is low and let T be such a tree that $T \subseteq [E]^{\preceq}$ and is two branching over E where $E = lvs(F \cup (\cup_{\sigma \in lvs(F^\sigma)} F^\sigma))$; moreover, for every $\sigma \in E$, there exists i_σ such that $T \cap [\sigma]^{\preceq} \subseteq \hat{C}^{-1}(i_\sigma)$. Since $\{F\}, \{F^\sigma\}_{\sigma \in lvs(F)}$, either there exists $\vec{\tau} \succeq lvs(F), \vec{\tau} \subseteq E$ such that $i_\tau = 0$ for all $\tau \in \vec{\tau}$ (progress on color 0); or there exists a $\hat{\sigma} \in lvs(F)$, a $\vec{\tau} \succeq lvs(F^{\hat{\sigma}})$ with $\vec{\tau} \subseteq E \cap [\hat{\sigma}]^{\preceq}$ such that $i_\tau = 1$ for all $\tau \in \vec{\tau}$ (progress on color 1).

Applications: Genearlized low solution

If we progress on color 0, then the next initial segment set is $\hat{F}, \hat{F}^\sigma, \sigma \in lvs(\hat{F})$ where $\hat{F}^\sigma = \emptyset$; if we progress on color 1, then the next initial segment set is $F, \hat{F}^\sigma$ where $\hat{F}^\sigma = F^\sigma$ if $\sigma \neq \hat{\sigma}$ and $\hat{F}^\sigma = \hat{F}$ if $\sigma = \hat{\sigma}$. In a word, a progress on color 0 can injure the current progress on color 1 but not vice versa.

Applications: Genearlized low solution

Note that either color 0 progress infinitely times which means we get a desired solution in color 0, or at some point color 0 no longer progress. Since $lvs(F)$ is finite, there must exists a $\hat{\sigma} \in lvs(F)$ such that color 1 progress infinitely times on $\hat{\sigma}$, which means we get a desired solution in color 1.

See [1] for more details about this section.

Definition 24

A problem P admit strong avoidance (resp. avoidance) of PA degree if for every P-instance (resp. computable P-instance) X there exists solution Y of X such that Y is not PA degree.

By CJS-decomposition of TT^2_k, to prove TT^2_k does not imply WKL_0, it suffices to prove the following:

Theorem 25

TT^1_k *admit strong avoidance of PA degree.*

Theorem 26

CTT^2_k *admit avoidance of PA degree.*

We firstly deal with Theorem 25.

Reduce to Π^0_1-class avoidance

Definition 27

A problem P admit Π^0_1-class avoidance of PA degree if for every non empty Π^0_1-class Q of P-instance, there exists a $X \in Q$ and a solution Y of X such that Y is not PA degree.

Obviously strong avoidance$\Rightarrow$ Π^0_1-class avoidance$\Rightarrow$ avoidance. But for strong avoidance of PA degree of TT^1_k, we have the following:

Theorem 28

TT^1_k admit strong avoidance of PA degree if and only if TT^1_k admit Π^0_1-class avoidance of PA degree.

Proof.

Combine the CJS-style Seetapun forcing with the Cross method of [8]. □

Reduce to Π^0_1-class avoidance

Moreover,

Theorem 29

TT^1_k *admit* Π^0_1*-class avoidance of PA degree.*

Remark 30

The notion of Π^0_1-class avoidance is useful in simplifying the proof of a result concerning WKL_0. It avoids the use of complicated injury method. Also note that Π^0_1-class avoidance notion for RT^1_2 give rise to the problem RWKL.

CJS-decomposition of CTT^2_k

Now we deal with CTT^2_k, namely Theorem 26. Note that the solution set of a given CTT^2_k instance C is not dense in the sense that not every finite perfect tree F extends to a solution of C. Therefore we firstly carry out the CJS-decomposition of CTT^2_k.

Given a tree $T \subseteq k^{<\omega}$, a $\rho \in T$, let $|\rho|_T$ denote the length of ρ with respect to T. Given a $\rho \in k^{<\omega}$, a set $B = \{b_0 < b_1 < \cdots < b_{n-1}\} \subseteq |\rho|$, let $\rho \restriction_B = \tau$ where $\tau(m) = \rho(b_m)$ for all $m \leq n-1$ and $|\tau| = |B|$.

CJS-decomposition of CTT^2_k

Definition 31

A *k-tree-split* on a tree T is a function $f : T \to \mathsf{Fin}(k^{<\omega})$ such that for every $\rho \in T$,

(i) $\emptyset \neq f(\rho) \subseteq k^{|\rho|_T}$;

(ii) $$\bigcup_{\rho' \succ \rho \wedge |\rho'|_T = |\rho|_T + 1} f(\rho') \restriction_{|\rho|_T - 1} = f(\rho);$$

Given a k-tree-split f on T, a tree $G \subseteq T$ is *homogeneous* for f if there exists an $f_G : G \to k^{<\omega}$ such that

(i) f_G is a homomorphism of G into $k^{<\omega}$ (i.e., for every $\rho, \rho' \in G$, $\rho \preceq \rho'$ implies $f_G(\rho) \preceq f_G(\rho')$) and $|f_G(\rho)| = |\rho|_G$;

(ii) For any $\rho \in G$, let $\rho_0 \prec \rho_1 \prec \cdots \prec \rho_n = \rho$ be the predecessors of ρ in G. Then there exists a $\zeta \in f(\rho)$ such that $f_G(\rho) = \zeta \restriction_{\{|\rho_0|_T - 1, |\rho_1|_T - 1, \cdots, |\rho_n|_T - 1\}}$.

CJS-decomposition of CTT^2_k

Definition 32

A k-tree-split f on T is *stable* iff for every $\sigma \in T$ there exists a $\rho \in T \cap [\sigma]^{\preceq}$ such that $f(\rho) \subseteq [\zeta]^{\preceq}$ for some ζ with $|\zeta| = |\sigma|_T$.

Proposition 33

For every weak stable coloring $C : [2^{<\omega}]^2 \to k$, there exists a Δ^0_2 k-tree-split f (on $2^{<\omega}$) such that for every tree G, if G is homogeneous for f, then C restricted on G is stable. Moreover, for every Δ^0_2 stable k-tree-split there exists a computable weak stable coloring $C : [2^{<\omega}]^2 \to k$ such that every homogeneous tree G of C is homogeneous for f.

In a word, Proposition 33 says that CTT^2_k can be CJS-decomposed into wCTT^2_k and k-tree-split.

Strong avoidance of k-tree-split

It's obvious that wCTT^2_k admit avoidance of PA degree. Meanwhile we have the following:

Theorem 34

For any k-tree-split f, there exists an infinite perfect tree G homogeneous for f such that G is not a PA degree.

The proof is by combining the cross method with the following property of k-tree-split.

Definition 35

A k-tree-split $f : T \to \mathsf{Fin}(k^{<\omega})$ is refined iff for every $\rho \in T$, every $\zeta \in f(\rho)$, there exists $\rho' \in T \cap [\rho]^{\preceq}$ such that $f(\rho'') \cap [\zeta]^{\preceq} \neq \emptyset$ for all $\rho'' \in T \cap [\rho']^{\preceq}$.

Strong avoidance of k-tree-split

Given two k-tree-split f, f', we have:

Lemma 36

1. *There exists a refined k-tree-split $\hat{f}$ (on T) such that $\hat{f}(\rho) \subseteq f(\rho)$ for all $\rho \in T$ (call such $\hat{f}$ a refinement of f).*
2. *If F is a finite tree homogeneous for a refinement $\hat{f}$ of f, then there exist a $\vec{\rho} \subseteq T$ with $\vec{\rho} \succeq \ell(F)$ and a k-tree-split $\tilde{f}$ on $T \cap [\vec{\rho}]^{\preceq}$ such that for every tree $G \subseteq T \cap [\vec{\rho}]^{\preceq}$ homogeneous for $\tilde{f}$, $F \cup G$ is homogeneous for f.*
3. *The collection of k-tree-split on T is a Π^0_1-class.*
4. *There exists a k^2-tree-split $\tilde{f}$ on T such that for every tree G homogenous for $\tilde{f}$, G is homogeneous for both f, f'.*

Non rigorously speaking, the sufficiency notion of Problem P is a sentence $\psi(\mathcal{F}, T, X)$ (usually Σ^0_1 or Π^0_1) about a finite set $\mathcal{F}$ of initial segment of P-solution, the associated Mathias tail T and a given P-instance X such that when $\psi(\mathcal{F}, T, X)$ is true, whatever the P-instance X looks like, one of the initial segment extends to a solution of X satisfying the restriction of T.

Example 37

Given a TT^1_k instance C. If $\mathcal{F}_0, \cdots, \mathcal{F}_{k-1}$ is a k-hierarchy with $\mathcal{F}_0 \neq \emptyset$, $\cup_{F\in\mathcal{F}_i} F \subseteq C^{-1}(i)$ and T is two branching over every $\sigma \in lvs(\cup_{i\in k, F\in\mathcal{F}_i} F)$, then whatever C looks like, there exists a $i \in k$, a $F \in \mathcal{F}_i$, an infinite tree $G \succeq F$ such that $G \setminus F \subseteq T \cap C^{-1}(i)$. i.e., the k-hierarchy gives the sufficiency notion of TT^1_k.

Example 38

Given a k-tree-split f. By Lemma 36 item 1,2, we have if for every k-tree-split f', there exists a $F \in \mathcal{F}$ homogeneous for f', then whatever f looks like, there exists a $F^* \in \mathcal{F}$, an infinite tree $G \succeq F^*$ (two branching over $lvs(F^*)$) such that G is homogeneous for f. By compactness argument (Lemma 36 item 3), this is a Σ^0_1 sentence.

Another well known example is that bushness gives the sufficiency notion for DNR. A tree T is 2-bushy if for every non leaf node $\sigma \in T$, there exists at least 2 immediate successors of σ in T.

Example 39

For every $X \in \omega^\omega$, every 2-bushy tree T, there exists $\sigma \in \ell(T)$ such that $\sigma(n) \neq X(n)$ for all $n \leq |\sigma|$. Which means σ extends to a DNR solution to X. Here a DNR solution to X is a $Y \in \omega^\omega$ such that $Y(n) \neq X(n)$ for all n.

The sufficiency notion plays a key role in all reverse math study of Ramsey type theorem (although sometimes it's too trivial to be noticed).

The sufficiency notion plays a key role in our low_2 construction of TT^2_k. To simplify things, we give a vague idea of our proof of low_2 construction of TT^2_k by demonstrating a proof of low_2 construction of CRT^2_2 using the sufficiency notion.

The first failed approach

Recall that the CJS-low_2 construction for CRT^2_2 is by defining a largeness notion. Fix a coloring $C : [\omega]^2 \to 2$. Let's firstly consider the following largeness notion (which fails).

Let $F, X \subseteq \omega$ where F is finite and $X > F$, let ψ be a Δ_0 formula. We say X is (F, ψ)-*large* iff: for every group of appropriate set $F_0, \cdots, F_{n-1}$ (i.e., $\cup_m F_m$ is finite and $\cup_m F_m \subseteq X$), every $x \in \omega$, there exists a $m \leq n-1$, an appropriate set G (i.e., $G \subseteq X \wedge G > F_m$) such that $|C_{F_m}(G)| = 1$ and $(\exists y)\psi(x, F \cup F_m \cup G \restriction_y)$. Where for a finite set $F = \{\sigma_0, \sigma_1, \cdots, \sigma_{s-1}\}$, $C_F(x) = C(\{\sigma_0, x\})^\frown \cdots {}^\frown C(\{\sigma_{s-1}, x\}) \in 2^F$.

Let's see why this largeness notion fails. The key point for a largeness notion to work is that the largeness notion should be preserved when shrinking the Mathias tail X in certain way (say in the way determined by C).

We *want* that for every appropriate group of finite sets $F_0, \cdots, F_{n-1}$, there exists an $m < n$, a color $\vec{i} \in 2^{F_m}$ of F_m such that $X \cap C_{F_m}^{-1}(\vec{i})$ is $(F \cup F_m, \psi)$-Large. Where $C_{F_m}^{-1}(\vec{i}) = \{z' : \forall z \in F_m[C(\{z, z'\}) = \vec{i}(z)]\ \}$.

However this is not true. Let's prove by contradiction and see where the proof fails. Suppose for every $m < n$, every color $\vec{i} \in 2^{F_m}$ of F_m, $X \cap C_{F_m}^{-1}(\vec{i})$ is not $(F \cup F_m, \psi)$-Large. So there is a sufficiently large $x \in \omega$ for every $m < n$, every $\vec{i} \in 2^{F_m}$, there exists an appropriate group of sets $F_{m,\vec{i},\hat{m}}, \hat{m} < n_{m,\vec{i}}$ such that for every appropriate G in the Mathias condition $(F \cup F_m \cup F_{m,\vec{i},\hat{m}}, X \cap C_{F_m}^{-1}(\vec{i}))$, if $|C_{F_{m,\vec{i},\hat{m}}}(G)| = 1$, then $\forall y \neg\psi(x, F \cup F_m \cup F_{m,\vec{i},\hat{m}}, G \upharpoonright y)$. Note that $F_m \cup F_{m,\vec{i},\hat{m}}, m < n, \vec{i} \in 2^{F_m}, \hat{m} < n_{m,\vec{i}}$ is an appropriate group. To derive a contradiction, we want that for every appropriate G, if G is monochromatic for $F_m \cup F_{m,\vec{i},\hat{m}}$, then $\forall y \neg\psi(x, F \cup F_m \cup F_{m,\vec{i},\hat{m}}, G \upharpoonright y)$. However, this is not true, we only have that if G is in color $\vec{i}$ of F_m and monochromatic for $F_m \cup F_{m,\vec{i},\hat{m}}$ then $\forall y \neg\psi(x, F \cup F_m \cup F_{m,\vec{i},\hat{m}}, G \upharpoonright y)$.

The second failed approach

In a word, each $F_m \cup F_{m,\vec{i},\hat{m}}$ only witnesses that ψ cannot be forced positively on *some* colors of F_m but not all of them. To remedy this let's try another largeness notion (which still fails).

X is (F,ψ)-*large* iff: for any appropriate set-color pairs $(F_0,\vec{i}_0),\cdots,(F_{n-1},\vec{i}_{n-1})$, there exists a $m<n$, an appropriate set G such that G is in color $\vec{i}_m$ of F_m and $(\exists y)\psi(F\cup F_m \cup G\upharpoonright_y)$.

This largeness notion fails because for some appropriate group of set-color pair $(F_0,\vec{i}_0),\cdots,(F_{n-1},\vec{i}_{n-1})$, there may not exists a $m<n$, such that $C^{-1}_{F_m}(\vec{i}_m)$ is non empty.

A largeness notion using sufficiency

To remedy this, we restrict on those group of sets who guarantee a $m < n$ such that $X \cap C^{-1}_{F_m}(\vec{i}_m)$ is infinite whatever C looks like.

A group of set-color pair $(F_0, \vec{i}_0), \cdots, (F_{n-1}, \vec{i}_{n-1})$ is *sufficient* iff for every function $g \in 2^{\cup_m F_m}$, there exists a $m < n$ such that $g \restriction_{F_m}$ agrees with $\vec{i}_m$. The intuition is that if a group of set-color pair is sufficient, then whatever C looks like, there must exists a $m < n$ such that $X \cap C^{-1}_{F_m}(\vec{i}_m)$ is infinite.

X is (F, ψ)-*large* iff: for any $x \in \omega$, any sufficient group of set-color pair $(F_0, \vec{i}_0), \cdots, (F_{n-1}, \vec{i}_{n-1})$, there exists a $m < n$, an appropriate set G such that G is in color $\vec{i}_m$ of F_m and $(\exists y)\psi(x, F \cup F_m \cup G \restriction_y)$.

Let's prove this largeness notion preserves. i.e., For any sufficient group of set-color pair $(F_0, \vec{i}_0), \cdots, (F_{n-1}, \vec{i}_{n-1})$, there exists a $m < n$, such that $X \cap C_{F_m}^{-1}(\vec{i}_m)$ is $(F \cup F_m, \psi)$-Large.

Suppose otherwise this is not true. Unfolding the definition, there exists an $x \in \omega$, and for each $m < n$, there exists an appropriate sufficient group of set-color pair $(F_{m,0}, \vec{i}_{m,0}), \cdots, (F_{m,n_m-1}, \vec{i}_{m,n_m-1})$ such that for every $\hat{m} < n_m$, every appropriate set G in color $\vec{i}_m, \vec{i}_{m,\hat{m}}$ of $F_m, F_{m,\hat{m}}$ respectively, we have: $(\forall y)\neg\psi(x, F \cup F_m \cup F_{m,\hat{m}} \cup G \restriction_y)$. Consider $(F_m \cup F_{m,\hat{m}}, \vec{i}_m \cdot \vec{i}_{m,\hat{m}}), m < n, \hat{m} < n_m$ as a group of set-color pairs. It suffices to prove that this group is sufficient.
Given a function $g \in 2^{(\cup_m F_m)\cup(\cup_{m,\hat{m}} F_{m,\hat{m}})}$, since $\cup_{m,\hat{m}} F_{m,\hat{m}}$ and $\cup_m F_m$ are mutually disjoint, g can be seen as a concatenation $g' \cdot \hat{g}$ where $g' = g \restriction_{\cup_m F_m}$ and $\hat{g} = g \restriction_{\cup_{m,\hat{m}} F_{m,\hat{m}}}$. Since $\{(F_m, \vec{i}_m)\}_{m<n}$ is sufficient, there exists a m' such that $g' \restriction_{F_{m'}} = \vec{i}_{m'}$; since $\{(F_{m',\hat{m}}, \vec{i}_{m',\hat{m}})\}_{\hat{m}<n_{m'}}$ is sufficient, there exists a $\hat{m} < n_{m'}$ such that $\hat{g} \restriction_{F_{m',\hat{m}}} = \vec{i}_{m',\hat{m}}$. Thus g agrees with $\vec{i}_{m'} \cdot \vec{i}_{m',\hat{m}}$ on $F_{m'} \cup F_{m',\hat{m}}$.

To make this approach works for wCTT^2_k, we need to incorporate combinatorics for trees. Because on trees, providing one witness $\{(F_{m,\hat{m}}, \vec{i}_{m,\hat{m}})\}_{\hat{m}<n_m}$ (for the largeness to fail) no longer guarantee that the resulted group is still sufficient.

Reverse math question

Question 40

Does STT^2_2 implies CTT^2_2?

Monin and Patey have recently make a breakthrough separating COH from SRT^2_2. Although their method likely applies here, the solutions of a CTT^2_2 instance is not dense. So we may simply separate the k-tree-split from STT^2_2. But we don't even know if RT^1_k (or even subset problem, fast growing function) could code (or not) k-tree-split.

Question 41

Does there exists a k-tree-split instance f such that for every $g \in \omega^\omega$, there exists a $\hat{g} \geq g$ such that $\hat{g}$ does not compute a solution of f? Does there exists a k-tree-split instance f and an infinite set X such that every subset $\hat{X}$ of X does not compute a solution of f.

Question 42

- What is the proof strength of "Ramsey degree of $A \leq r$" with respect to $\mathcal{U}$ (hence forth $\mathsf{RD}(A, r; \mathcal{U})$).
- For $r' > r$, show that for some structure $\mathcal{U}$, $\mathsf{RD}(A, r'; \mathcal{U})$ is weaker than $\mathsf{RD}(A, r; \mathcal{U})$.

Devlin proved that pairs have Ramsey degree 2 with respect to $(\mathbb{Q}, <)$ (hence forth $\mathsf{RD}(2, 2; (\mathbb{Q}, <))$) and we have noticed that $\mathsf{RD}(2, 3; (\mathbb{Q}, <))$ is equivalent to ACA. But we guess $\mathsf{RD}(2, 4; (\mathbb{Q}, <))$ does not.

Question 43

Is it true that $\mathsf{RD}(2, 4; (\mathbb{Q}, <))$ does not imply ACA?

Usually, to prove $\mathsf{RD}(A, r; \mathcal{U})$ means to find a structure satisfying a particular picture. For example, $\mathsf{RD}(2, 2; (\mathbb{Q}, <))$ is proved by showing that for every coloring $C : [\mathbb{Q}]^2 \to k$, there exists a $(X, <) \cong (\mathbb{Q}, <)$ and $i_0, i_1 \in k$ such that

$$\forall x \in X \forall^\infty y \in (x, \infty) \cap X[C(\{x, y\}) = i_0];$$
$$\forall x \in X \forall^\infty y \in (-\infty, x) \cap X[C(\{x, y\}) = i_1].$$

Since $\mathsf{RD}(2, 2; (\mathbb{Q}, <))$ is equivalent to ACA, to prove $\mathsf{RD}(2, 2; (\mathbb{Q}, <))$, it is necessary to construct an X as above. But on general, we do not know

Question 44

Is this always the case?

Unlike the classical Ramsey theorem where RT^{d+1}_k implies RT^d_k, it is not obvious whether $\mathsf{RD}(d+1, r; (\mathbb{Q}, <))$ implies $\mathsf{RD}(d, r'; (\mathbb{Q}, <))$ when $r' < r$. Analyzing the proof of Devlin's theorem (see Section 6.3 of [12]), it seems still true that $\mathsf{RD}(d, r_d; (\mathbb{Q}, <))$ is equivalent to ACA when $d \geq 2$. Where r_d refers to the Ramsey degree of d-tuple with respect to $(\mathbb{Q}, <)$. But the general case is not clear.

Question 45

Given two finitely generated substructure $A \subseteq A'$ of $\mathcal{U}$, does $\mathsf{RD}(A', r_{A'}; \mathcal{U})$ implies $\mathsf{RD}(A, r_A; \mathcal{U})$. Where $r_A, r_{A'}$ refers to the Ramsey degree of A, A' respectively with respect to $\mathcal{U}$.

Others

Question 46

Is there an abstract definition of (an automatic way to derive) CJS-decomposition?

Our current understanding of CJS-decomposition of a problem P is following. $\mathsf{P} = \mathsf{S} + \mathsf{C}$. Where solutions of C instance is "dense" (in the sense that every initial segment extends to a solution). The collection of S instance is a Π^0_1 class; given a S instance X, the "finite" solution of X is enumerable in X, the set of solutions of X is a $\Pi^{0,X}_1$ class.

Question 47

Given a problem P, is there a "most" hyperimmune property for P?

For example, RT^1_2 instance $X : \omega \to 2$ is 2-hyperimmune if for every computable array of increasing disjoint finite set pairs $\{(A^n_0, A^n_1)\}_{n\in\omega}$, there exists an n^* such that $A^{n^*}_i \subseteq X^{-1}(i)$; X is hyperimmune if for every $i \in 2$, every computable array of increasing finite set $\{A_n\}_{n\in\omega}$, there exists an n^* such that $A_{n^*} \subseteq X^{-1}(i)$. So 2-hyperimmune implies hyperimmune. And 2-hyperimmune seems to be the strongest known hyperimmune property of RT^1_2.

A hyperimmune property of a P-instance X usually says that for every computable list of description of X, X diagonalize one of them whenever it is possible to do so. Clearly, we need to put certain restriction on the description allowed. The property usually makes it hard to solve X, but can also imply existence of weak solutions (For example, Denis pointed to me that if color i of a Δ^0_2 RT^1_2 instance is hyperimmune, then it admit a *low* solution in color $1-i$).

As far as I could think of, there is no two known hyperimmune properties of a problem contradicting each other. This indicates that there might be a "most" hyperimmune property for many problems.

Thank you for attending. Is there any questions?

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