Continuous Model Theory Revisited.
Association for Symbolic Logic.

H. Jerome Keisler

March 26, 2020
1. Overview of Continuous Model Theory

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**Metric structures:** General structures with extra requirements—a distinguished metric, uniformly continuous functions and predicates.

A highly developed model theory parallel to first order model theory.
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Metric structures / First order structures with equality

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- **Analogy**
  
  *Metric structures / First order structures with equality*
  
  *General structures / First order structures without equality.*

- **Punch Line**
  
  *Almost all of the model theory for metric structures carries over in a precise way to general $[0, 1]$-valued structures.*
2. General Structures

Syntax:

Vocabulary $V$: predicate, function, and constant symbols.

In this talk, $V$ is always countable.
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Truth values: $[0, 1]$, $0 = \text{True}$, $1 = \text{False}$. 
Variables: $x_0, x_1, \ldots$ 
Connectives: continuous functions $C : [0, 1]^n \to [0, 1]$. 
Quantifiers: $\text{sup}, \text{inf}$. 
Terms, atomic formulas: as in first order logic. 
Formulas, sentences: built using connectives and quantifiers.
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General structure $\mathcal{M}$ with vocabulary $V$ and universe $M$:

$F^\mathcal{M} : M^n \rightarrow M$ for each $n$-ary function symbol $F \in V$.
$P^\mathcal{M} : M^n \rightarrow [0, 1]$ for each $n$-ary predicate symbol $P \in V$.
$c^\mathcal{M} \in M$ for each constant symbol in $c \in V$.
$\varphi^\mathcal{M} : M^{\lvert \overrightarrow{x} \rvert} \rightarrow [0, 1]$ is defined inductively on formulas $\varphi(\overrightarrow{x})$. 
3. Pre-metric and Metric Structures

[Ben Yaacov, Berenstein, Henson, Usvyatzev, 2008]

Signature $L$: A vocabulary equipped with distinguished binary predicate symbol $d$ (for distance) and a modulus of uniform continuity for each function and predicate symbol.

Pre-metric structure $M$ with signature $L$:
General structure where $d_M$ is a pseudo-metric, and for each symbol $S \in V$, $S_M$ is uniformly continuous with respect to $d_M$.

(More complicated than general structures.)

Follows that each $\phi_M$ ($\cdot$) is uniformly continuous w.r.t. $d_M$.

Metric theory: A set of sentences $U$ equipped with a signature $L$ such that every general model of $U$ is a pre-metric structure.

Metric structure: Pre-metric structure where $d_M$ is a complete metric.

Every pre-metric structure $M$ has a unique completion $M^\equiv = M$.

Canonical example: Unit ball of a Banach space.
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4. Basic Model Theory for General Structures

A general theory $T$ is a set of sentences.

$\varphi^M$ denotes truth value in $[0, 1]$ of sentence $\varphi$ in $M$.

$M \models T$ means $\varphi^M = 0$ for all $\varphi \in T$. 

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The following are defined as usual:
$M \equiv N$, $M \prec N$, $Th(M)$.

Type of $b$ over $A$: $tp_M(b/A) = Th(M, \{b\} \cup A)$.

$M$ is $\lambda$-saturated if for every $A \subseteq M$ of size $< \lambda$,
every type over $A$ realized in some $N \succ M$ is realized in $M$. 
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**Reduction** of $M$: Identify $a, b$ if $(M, a, \bar{x}) \equiv (M, b, \bar{x})$ for all $\bar{x} \subseteq M$.

If $M, N$ are reduced, $M \cong N$ says they are isomorphic.
4. Basic Model Theory for General Structures


A **general theory** \( T \) is a set of sentences. 
\( \varphi^\mathcal{M} \) denotes truth value in \([0, 1]\) of sentence \( \varphi \) in \( \mathcal{M} \). 
\( \mathcal{M} \models T \) means \( \varphi^\mathcal{M} = 0 \) for all \( \varphi \in T \).

The following are defined as usual: 
\( \mathcal{M} \equiv \mathcal{N} \), \( \mathcal{M} \prec \mathcal{N} \), \( \text{Th}(\mathcal{M}) \).

**Type** of \( b \) over \( A \): \( \text{tp}_{\mathcal{M}}(b/A) = \text{Th}(\mathcal{M}, \{b\} \cup A) \).

\( \mathcal{M} \) is **\( \lambda \)-saturated** if for every \( A \subseteq M \) of size \( < \lambda \), every type over \( A \) realized in some \( \mathcal{N} \succ \mathcal{M} \) is realized in \( \mathcal{M} \).

**Reduction** of \( \mathcal{M} \): Identify \( a, b \) if \( (\mathcal{M}, a, \vec{x}) \equiv (\mathcal{M}, b, \vec{x}) \) for all \( \vec{x} \subseteq M \). 
If \( \mathcal{M}, \mathcal{N} \) are reduced, \( \mathcal{M} \cong \mathcal{N} \) says they are isomorphic.

**Ultraproducts** constructed using reduction.

**Compactness Theorem** proved using ultraproducts.

**Monster structure**: Reduced and \( \kappa \)-saturated of inaccessible size \( \kappa > \aleph_0 \).

**Small** means of cardinality \( < \kappa \).
5. Definable Predicates

Let $T$ be a general theory.

**Definition**

A sequence of formulas $\langle \varphi_k(\vec{x}) \rangle_{k \in \mathbb{N}}$ is **Cauchy** in $T$ if

$$(\forall \varepsilon > 0)(\exists m)(\forall k \geq m) \models \sup_{\vec{x}} |\varphi_m(\vec{x}) - \varphi_k(\vec{x})| \leq \varepsilon.$$
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If $\langle \varphi_k(\vec{x}) \rangle_{k \in \mathbb{N}}$ is Cauchy in $T$, then for each $M \models T$, we write

$$[\lim \varphi_k]_M(\cdot) = \lim_{k \to \infty} \varphi_k^M(\cdot).$$

This limit always exists. $[\lim \varphi_k]_M$ maps $M|\vec{x}|$ into $[0, 1]$, and is called a **definable predicate** in $M$. 
6. The Main Definition: Pre-metric Expansion

Let $V$ be a vocabulary, $V_D = V \cup \{D\}$, $D$ a binary predicate symbol, $T$ a general theory with vocabulary $V$. 

Definition $T$ is a pre-metric expansion of $T$ if:

(i) $T$ is a metric theory whose signature $L$ is over $V_D$ with distance $D$.

(ii) There is a sequence $\langle d \rangle = \langle d^k(x,y) \rangle_{k \in \mathbb{N}}$ of $V$-formulas Cauchy in $T$ such that the general models of $T$ are exactly the structures $M_e = (M,\lim d^k_M)$ where $M|_T = T$.

$M_e$ is called the pre-metric expansion of $M$ for $T$.

$\langle d \rangle$ is called an approximate distance for $T$. (Not unique).

Note that $D_{M_e}$ is a definable predicate in $M$ (defined by $\langle d \rangle$).
Let \( V \) be a vocabulary, \( V_D = V \cup \{ D \} \), \( D \) a binary predicate symbol, \( T \) a general theory with vocabulary \( V \).

**Definition**

\( T_e \) is a **pre-metric expansion** of \( T \) if:

(i) \( T_e \) is a metric theory whose signature \( L_e \) is over \( V_D \) with distance \( D \).

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$\mathcal{M}_e$ is called the **pre-metric expansion** of $\mathcal{M}$ for $T_e$. $\langle d \rangle$ is called an **approximate distance** for $T_e$. (Not unique).

Note that $D^\mathcal{M}_e$ is a definable predicate in $\mathcal{M}$ (defined by $\langle d \rangle$).
7. The Metric Expansion Theorem

Theorem

*Every general theory has a pre-metric expansion.*
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Stronger form:

**Theorem**

*Every general theory $T$ has a pre-metric expansion $T_e$ with an approximate distance $\langle d_k \rangle_{k \in \mathbb{N}}$ such that $d_k^M$ is a pseudo-metric for every $k \in \mathbb{N}$ and $M \models T$.*
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These results have far-reaching consequences, which extend most of the model theory for metric structures to general structures.
A **property** is a class of structures closed under isomorphism.

**Definition**

A property $\mathcal{P}$ of general structures is **absolute** if for every general structure $\mathcal{M}$ and pre-metric expansion $\mathcal{M}_e$, $\mathcal{M}$ has property $\mathcal{P}$ if and only if $\mathcal{M}_e$ has property $\mathcal{P}$.

Trivial Example: For each $\mathcal{V}$-formula $\phi(\vec{x})$ and tuple $\vec{a}$ of parameters, the property $\mathcal{M}|=\phi(\vec{a})$ is absolute.
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If $A$ is a set of new constant symbols, then every pre-metric expansion of $T$ as a $V$-theory is also a pre-metric expansion of $T$ as a $(V \cup A)$-theory.
8. Absoluteness

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Since \( M \) and \( M_e \) have the same universe set \( M \), we can consider absoluteness of properties with extra parameters from \( M \).
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**Trivial Example**: For each $V$-formula $\varphi(\vec{x})$ and tuple $\vec{a}$ of parameters, the property $\mathcal{M} \models \varphi(\vec{a})$ is absolute.
The property of being reduced is absolute.
Proposition

The property of being reduced is absolute.

The property of a mapping $P : M^n \rightarrow [0, 1]$ being a definable predicate is absolute.

That is, $P$ is definable in $\mathcal{M}$ iff $P$ is definable in $\mathcal{M}_e$.

Being an elementary substructure is absolute.

That is, if $M, N \models T$, then $M \preceq N$ iff $M_e \preceq N_e$.

And $b$ and $c$ having the same type over $A$ is absolute.

Being $\lambda$-saturated is absolute.

Being a monster structure is absolute.
9. Some Absolute Properties of General Structures

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10. Absolute Version of a Pre-metric Property

**Definition**

A property $\mathcal{P}$ of general structures is an **absolute version** of a property $\mathcal{Q}$ of pre-metric structures if $\mathcal{P}$ is absolute and agrees with $\mathcal{Q}$ on pre-metric structures.

**Corollary** (of the Metric Expansion Theorem)

Every property $\mathcal{Q}$ of pre-metric structures has $\leq 1$ absolute version.

**Proof.**

Suppose $\mathcal{P}_1, \mathcal{P}_2$ are absolute versions of $\mathcal{Q}$. Consider a general $\mathcal{M}$. By the Metric Expansion Theorem, $\mathcal{M}$ has a pre-metric expansion $\mathcal{M}_e$. Then $\mathcal{P}_1(\mathcal{M})$ iff $\mathcal{P}_1(\mathcal{M}_e)$ iff $\mathcal{Q}(\mathcal{M}_e)$ iff $\mathcal{P}_2(\mathcal{M}_e)$ iff $\mathcal{P}_2(\mathcal{M})$. 

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11. From Pre-metric to General Properties

If a property $Q$ of pre-metric structures has an absolute version $P$, we consider $P$ to be the “right” extension of $Q$ to general structures.
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We say that a general structure has a pre-metric property $Q$ if it satisfies the absolute version of $Q$. 

Most of the main properties of pre-metric structures in the literature have absolute versions that can be characterized in terms of $M$ itself without mentioning pre-metric expansions. (For example, $\text{Th}(M)$ being stable, simple, or rosy).

Plan: Build a library of such characterizations of absolute versions.

Some Properties Without Absolute Versions:

- $D$ has diameter one.
- $\phi(M)$ is Lipschitz continuous with respect to $D$. 

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12. Topological and Uniform Properties

**Proposition**

A set $S \subseteq M^n$ being closed has an absolute version. 

$S$ is closed iff there is a set $\Phi(\vec{x})$ of $V$-formulas such that

$$S = \{ \vec{b} \in M^k : M \models \Phi(\vec{b}) \}.$$
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**Corollary**

A set $S \subseteq M^n$ being compact has an absolute version. $S_0$ being dense in $S$ has an absolute version. The property that $\langle b_k \rangle$ converges to $c$ has an absolute version.
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The property that $\langle b_k \rangle$ converges to $c$ has an absolute version.

**Proposition**

A sequence of elements $\langle b_k \rangle$ being Cauchy has an absolute version.

$\langle b_k \rangle$ is Cauchy in $\mathcal{M}$ iff it has a limit in some $\mathcal{N} \succ \mathcal{M}$. 
Corollary

Being a complete structure has an absolute version. So
Corollary

*Being a complete structure has an absolute version. So $\mathcal{M}$ is complete iff every pre-metric expansion of $\mathcal{M}$ is a metric structure.*
13. Completions of General Structures

Corollary

Being a complete structure has an absolute version. So $\mathcal{M}$ is complete iff every pre-metric expansion of $\mathcal{M}$ is a metric structure. $\mathcal{M}$ is complete iff it is reduced and each Cauchy $\langle b_k \rangle$ has a limit in $\mathcal{M}$.
Corollary

Being a complete structure has an absolute version. So $\mathcal{M}$ is complete iff every pre-metric expansion of $\mathcal{M}$ is a metric structure. $\mathcal{M}$ is complete iff it is reduced and each Cauchy $\langle b_k \rangle$ has a limit in $\mathcal{M}$.

Definition

A general structure $\mathcal{M}$ is a completion of $\mathcal{N}$ if $\mathcal{M}$ is complete and the reduction of $\mathcal{N}$ is a dense elementary substructure of $\mathcal{M}$.

Corollary

Every general structure has a unique completion up to isomorphism.
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Being a complete structure has an absolute version. So \( M \) is complete iff every pre-metric expansion of \( M \) is a metric structure. \( M \) is complete iff it is reduced and each Cauchy \( \langle b_k \rangle \) has a limit in \( M \).

Definition

A general structure \( M \) is a **completion** of \( N \) if \( M \) is complete and the reduction of \( N \) is a dense elementary substructure of \( M \).

Corollary

Every general structure has a unique completion up to isomorphism.

Proposition

Every \( \aleph_1 \)-saturated reduced general structure is complete. So every monster structure is complete.
In a metric structure, a set $S$ is **definable over** $A$ if $S$ is closed and $\text{dist}(x, S) = \inf\{D(x, y) : y \in S\}$ is a definable predicate over $A$.

$b \in \text{dcl}(A)$ if $\{b\}$ is definable over $A$.

$b \in \text{acl}(A)$ if $b \in C$ for some compact $C$ definable over $A$. 

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**Proposition**

Being a definable set over $A$ has an absolute version.

$S$ is definable over $A$ iff $S$ is closed and for each $V$-formula $\varphi^M(x, y)$, if $\varphi^M$ is a pseudo-metric then $\text{dist}_\varphi(x, S) = \inf \{\varphi(x, y) : y \in S\}$ is a definable predicate over $A$.
14. Definable and Algebraic Closure

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**Proposition**

$b \in \text{dcl}(A)$ and $b \in \text{acl}(A)$ have absolute versions.

Let $\mathcal{M}$ be reduced and $\aleph_1$-saturated.

$b \in \text{dcl}(A)$ iff $b$ is the only realization of $\text{tp}(b/A)$ in $\mathcal{M}$.

$b \in \text{acl}(A)$ iff the set $\{c : \text{tp}(c/A) = \text{tp}(b/A)\}$ is compact in $\mathcal{M}$. 
15. Stable Theories

Definition

A complete general theory $T$ with monster model $\mathcal{M}$ is stable if there is a small cardinal $\lambda < |\mathcal{M}|$ such that whenever $A \subseteq \mathcal{M}$ and $|A| \leq \lambda$, the set of complete types over $A$ in $\mathcal{M}$ has cardinality $\leq \lambda$. 

Corollary

Being stable is absolute.

A stable independence relation is a ternary relation on small sets that satisfies Invariance, Symmetry, Transitivity, Finite Character, Full Existence, Local Character, and Stationarity.

Theorem

A complete general theory $T$ is stable iff the monster model of $T$ has a (unique) stable independence relation.
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16. Comparing Theories via Ultrapowers

Let $\mathcal{M}, \mathcal{N}$ be general structures, $T, U$ be complete continuous theories.

**Definition**

$D$ **saturates** $\mathcal{M}$ if $D$ is a regular ultrafilter over a set $I$, and the ultrapower $\mathcal{M}^I/D$ is $|I|^+$-saturated.
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- $\mathcal{D}$ **saturates** $\mathcal{M}$ if $\mathcal{D}$ is a regular ultrafilter over a set $I$, and the ultrapower $\mathcal{M}^I/\mathcal{D}$ is $|I|^+$-saturated.
- $\mathcal{M} \preceq \mathcal{N}$ if every $\mathcal{D}$ that saturates $\mathcal{N}$ saturates $\mathcal{M}$.
- $\mathcal{M} \triangleleft \mathcal{N}$ if $\mathcal{M} \preceq \mathcal{N}$ but not $\mathcal{N} \preceq \mathcal{M}$.
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If $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N} \preceq \mathcal{M}$.

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$T \triangleleft U$ means “$\mathcal{M} \models T \land \mathcal{N} \models U \Rightarrow \mathcal{M} \triangleleft \mathcal{N}$”.

$T \triangleleft = \{U: T \triangleleft U \text{ and } U \triangleleft T\}$.

$\mathcal{G}$ is the set of all $T \triangleleft$. $(\mathcal{G}, \triangleleft)$ is a partial ordering.
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**Projects:** Study $(\mathcal{G}, \preceq)$. Use $\preceq$ to classify structures.
17. Known Results for first-order (FO) Theories

Let $\mathcal{F} = \{ T \models \subseteq : T \text{ is first order} \}$. Let $T^{rg} = Th(\text{random graph})$.
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Let $\mathbb{F} = \{ T \sqsubseteq: T \text{ is first order}\}$. Let $T^\text{rg} = \text{Th}(\text{random graph})$. On this page, $T, U$ denote complete FO theories.

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**Shelah 1972**: Let $\text{stb}_\mathbb{F} = \{ T \sqsubseteq: T \text{ stable but not } \sqsubseteq\text{-minimal}\}$.

$\text{stb}_\mathbb{F} \in \mathbb{F}$. For every unstable $U$, $\text{min}_\mathbb{F} \sqsubseteq \text{stb}_\mathbb{F} \sqsubseteq U \sqsubseteq$.

**Malliaris 2012**: $T^\text{rg}$ is $\sqsubseteq\text{-minimal among unstable theories}$.

If $U \sqsubseteq T^\text{rg}$ then $U$ is simple.

**Malliaris-Shelah, 2016–2019**: Every $\text{SOP}_2$ theory is $\sqsubseteq\text{-maximal}$ (converse is a conjecture).

$\text{H. Jerome Keisler}$

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Let $\mathcal{F} = \{ T \trianglelefteq: T \text{ is first order} \}$. Let $T^{rg} = Th(\text{random graph})$.
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$\text{stb}_\mathcal{F} \in \mathcal{F}$. For every unstable $U$, $\text{min}_\mathcal{F} \vartriangleleft \text{stb}_\mathcal{F} \vartriangleleft U \trianglelefteq$.

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If $U \trianglelefteq T^{rg}$ then $U$ is simple. $T^{rg}$ is not $\trianglelefteq$-maximal.

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Every $SOP_2$ theory is $\trianglelefteq$-maximal (converse is a conjecture).

$(\mathcal{F}, \trianglelefteq)$ restricted to simple FO theories is extremely rich.
It contains a copy of $(\mathcal{P}(\mathbb{N}), \subseteq)$.

Suppose there is a super-compact cardinal.
If $T$ is simple and $U \trianglelefteq T$, then $U$ is simple.
On this page, $T, U$ denote complete general theories. There is a natural embedding $h: (F, \sqsubseteq) \to (G, \sqsubseteq)$.
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There is a natural embedding $h: (F, \sqsubseteq) \rightarrow (G, \sqsubseteq)$.

**Theorem**

$\mathcal{M} \sqsubseteq \mathcal{N}$ is absolute.

*For any pre-metric expansion $T_e$ of $T$, $T \sqsubseteq T_e$ and $T_e \sqsubseteq T$.***
On this page, $T$, $U$ denote complete general theories. There is a natural embedding $h: (F, ≪) → (G, ≪)$.

**Theorem**

$M ≪ N$ is absolute. 
For any pre-metric expansion $T_e$ of $T$, $T ≪ T_e$ and $T_e ≪ T$.

Let $stb_G$ be the class of stable theories that are not $≪$-minimal.

**Theorem**

$stb_G$ belongs to $G$. 
18. The First Two Classes in $(G, \triangleleft)$

On this page, $T, U$ denote complete general theories. There is a natural embedding $h: (F, \triangleleft) \rightarrow (G, \triangleleft)$.

**Theorem**

$M \triangleleft N$ is absolute.

For any pre-metric expansion $T_e$ of $T$, $T \triangleleft T_e$ and $T_e \triangleleft T$.

Let $stb_G$ be the class of stable theories that are not $\triangleleft$-minimal.

**Theorem**

$stb_G$ belongs to $G$. For every unstable $U$, $\text{min}_G \triangleleft stb_G \triangleleft U_\triangleright$. 
18. The First Two Classes in \((G, \preceq)\)

On this page, \(T, U\) denote complete general theories. There is a natural embedding \(h: (F, \preceq) \to (G, \preceq)\).

**Theorem**

\(M \preceq N\) is absolute.  
*For any pre-metric expansion \(T_e\) of \(T\), \(T \preceq T_e\) and \(T_e \preceq T\).*

Let \(stb_G\) be the class of stable theories that are not \(\preceq\)-minimal.

**Theorem**

\(stb_G\) belongs to \(G\). *For every unstable \(U\), \(\text{min}_G \preceq \text{stb}_G \preceq U\).*

**Question**

*Given a complete continuous theory \(T\), is there a FO theory \(T_0\) such that \(T \preceq T_0\) and \(T_0 \preceq T\)? Is \(h: (F, \preceq) \to (G, \preceq)\) onto?*
References


