(EXTRA)ORDINARY EQUIVALENCES WITH ADS

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**Theorem (Rival Sands)**

Let \((P, <_P)\) be an infinite poset of finite width. Then there exists an infinite chain \(C\) such that every element of \(P\) is comparable to none or to infinitely many elements of \(C\). Moreover, if \(P\) is countable, \(C\) may be chosen so that each \(p \in P\) is comparable to none or to cofinitely many elements of \(C\).

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I. Rival, B. Sands

On the adjacency of vertices to the vertices of an infinite subgraph.


A poset has **finite width** if there exists \(k \in \mathbb{N}\) such that each antichain in \(P\) has size at most \(k\).
In order to analyse the previous theorem in reverse mathematics it is convenient to introduce these families of principles

<table>
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<tr>
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Ramsey-theoretic principle?

Three measures of proximity of Rival-Sands theorem and Ramsey’s theorem

1. Rival and Sands present their theorems as trade-off of Ramsey’s theorem for pairs $RT_2^2$

2. $sRSpo_\kappa$ is a Ramsey-type statement

3. we prove some equivalences with ADS, the ascending/descending sequence principle, a consequence of Ramsey’s theorem for pairs
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For each countable graph there exists an infinite subgraph $H$ such that which is either complete or totally disconnected

- instances of $\text{RSpo}_k$ and $\text{sRSpo}_k$, i.e. countable comparability graphs such that the size of their totally disconnected subgraphs is bounded by some $k$, form a subclass of instances of $\text{RT}_2$

  $\text{RSpo}_k$ is a (combinatorial) improvement of a more general statement proved by Rival and Sands in the same paper, whose instances are countable graphs

- solutions to $\text{RSpo}_k$ and $\text{sRSpo}_k$ form a subclass of the solutions of $\text{RT}_2$ (even restricted to instances of $\text{RSpo}_k$ and $\text{sRSpo}_k$)

  solutions to $\text{RSpo}_k$ and $\text{sRSpo}_k$ give information about the adjacency relation between the inside and the outside of the solution. While solutions to $\text{RT}_2$ carry information about the adjacency relation only of the inside
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A statement of the form $\forall G(\varphi(G) \Rightarrow \exists H\psi(G, H))$ is said to be of Ramsey-type when it has the following properties:

- if $\varphi(G)$ and $\psi(G, H)$, then $H$ must be infinite,
- if $\varphi(G)$, $\psi(G, H)$, and $H' \subseteq H$ is infinite, then also $\psi(G, H')$.

$RSpo_k$ is NOT in general a Ramsey-type principle.

$sRSpo_k$ is a Ramsey-type principle: let $(P, <_P)$ be a poset of width $k$, $C$ a solution and $D \subseteq C$ be infinite. If $p \in P$ is comparable with only finitely many elements of $D$, then $p$ is comparable with some elements of $C$ and incomparable with infinitely elements of $C$, contrary to the fact that $C$ is a solution.

$sRSpo_k$ Each countable poset $(P, <_P)$ of width $k$ contains an infinite chain $C$ such that every point in $P$ is comparable to none or to CO-FINITELY many elements in $C$. 
Ramsey-theoretic principle?

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Each countable linear order \((L, <_L)\) contains an \(\omega\) or an \(\omega^*\) chain.

**Theorem**

Over \(\text{RCA}_0\), for each \(k \geq 3\), \(\text{ADS}\) is equivalent to \(\text{RSpo}_k\).

- \(\text{RSpo}_k\) is the first theorem from ordinary mathematics proved to be equivalent to \(\text{ADS}\).
- Our result improves considerably the upper bound, \(\Pi^1_1 - \text{CA}_0\), given by the original proof of \(\text{RSpo}_k\).
- We gave proofs of \(\text{RSpo}_k\) which are very different combinatorially from the original proof.
Key elements of the proof

1. Chain decomposition

   Kierstead proved that for each $k$ and for each computable poset $(P, <_P)$ of width $k$, there exist $h \leq 5^k$ and $C_0, \ldots, C_h$ computable chains such that $P = \bigcup_{i < h} C_i$. His proof can be formalised in RCA$_0$

2. ADS is used only once at the very beginning of the proof to get an ascending or a descending chain $A$

3. starting from $A$, searching iteratively witnesses of the fact that ascending or descending chains are not solutions to $\text{RSp}_{k'}$, we get a solution to $\text{RSp}_{k'}$
**Theorem**

Over WKL$_0$, SADS is equivalent to RSpo$_2$

WKL$_0$ is used in the proof only to decompose a poset of width two into two chains. Hence, over RCA$_0$, SADS is equivalent to the following variant of RSpo$_2$, RSpo$_2^{\text{Dec}}$:

for each countable poset $(P, <_P)$ such that there exist two chains $D$ and $E$ such that $P = D \cup E$, there exists an infinite chain $C$ such that every point in $P$ is comparable to none or to infinitely many elements in $C$. 
**Theorem**

Over RCA₀, ADS is equivalent to sRSpo₂

Thus, sRSpo₂ is strictly stronger than RSpo₂^{Dec}
Theorem
Over $\text{RCA}_0$, $\text{sRSpo}_2$ implies $\text{ADS}$

Proof.
Let $(L, \leq_L)$ be a linear order and let $P = (L \times 2, \leq_P)$ the order on the Cartesian product of $L$, so that $(\ell, i) \leq_P (m, j) \iff \ell \leq_L m \land i \leq j$. Such a poset has clearly width two, so let $C \subseteq P$ be a solution. For each $i < 2$ set $C_i = C \cap (L \times i)$. We claim that if $C_0$ is infinite, then $C_0$ has no minimum, and can thus be refined to a descending chain. Suppose on the contrary that $C_0$ is infinite and that $(m, 0)$ is minimum in $C_0$. By definition of $\leq_P$ it holds that $(m, 0) <_P (m, 1)$ and $(n, 0) |_P (m, 1)$, for each $n >_L m$. It follows that $(m, 1)$ is incomparable with infinitely many elements of $C$, contrary to the assumption that $C$ is a solution.

Similar reasoning allows us to prove that if $C_1$ is infinite, then $C_1$ has no maximum, and hence that $L$ contains an ascending chain. \qed
Counterexample to an $\omega$ chain

Let $(P, <_P)$ be a poset of width two.
Let $A$ be an $\omega$ chain whose tails are not solutions.
There exists an $\omega$ chain $B$ such that each $b \in B$ is above some $a \in A$ and incomparable with a tail of $A$.

- since $P$ has width two it is enough to look at $B = \langle b_n \mid n \in \mathbb{N} \rangle$ such that
  $$\forall n \exists m > n \exists \ell > n (a_m <_P b_\ell \land a_{m+1} \nless_P b_\ell)$$

  We call such $B$ a counterexample to $A$

- if $P$ has width two there exists $f: \mathbb{N} \to \mathbb{N}$ which enumerates $B$. This $f$ allows to find counterexamples to ascending chains uniformly.
Theorem
Over $\text{RCA}_0$, ADS implies $\text{sRSpo}_2$.

Idea of the proof.
Let $(P, <_P)$ be a poset of width two. Assume that $P$ does not have any solution to $\text{sRSpo}_2$. Assume $A_0$ is an ascending chain in $P$ (if $A_0$ is descending consider $(P, >_P)$).

Since $A_0$ is not a solution, there exists a counterexample $A_1$. Since $A_1$ is not a solution, so there exists a counterexample $A_2$. And so on. We thus get a sequence $\langle A_n \mid n \in \mathbb{N} \rangle$ of ascending chains as in the picture.

To make sure the sequence itself exists in $\text{RCA}_0$ we define a uniform procedure to define it. To this end it is crucial that for each $n$ and $m$ the $m^{th}$ element of $A_n$ can be chosen after inspecting only initials segments of $A_0, \ldots, A_{n-1}$.

Let $S$ be an ascending chain with one element from each $A_n$. Then $S$ is a solution: each element of $P$ is comparable with co-finitely many elements of $S$. \qed
**Question**
What is the strength of sRSpo$_k$ for $k \geq 2$?

$\Pi^1_1 - CA_0$ is the upper bound and ADS the lower bound.

The proof for sRSpo$_2$ in ADS exploits crucially the fact that the poset has width two.