Results in Computable Model Theory of Continuous Logic

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Plan

(i) Preliminaries on Continuous Logic.

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(ii) Effectivizing Continuous Logic.

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(iii) Effective Completeness Theorem

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(i) Preliminaries on Continuous Logic.

(ii) Effectivizing Continuous Logic.

(iii) Effective Completeness Theorem

(iv) Computational vs. Syntactic Complexity

Preliminaries on Continuous Logic

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$\neg(P(x_0) \wedge (\neg P(x_0)))$

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But representing continuous structures in this setting can be tedious, and sometimes impossible.

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Binary predicate symbol: $\{d(\cdot,\cdot)\}$

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Unlike classical logic, since 0 represents truth and 1 falsehood, $\vee$ interpreted as the maximum operator acts as an 'and' gate, and $\wedge$ interpreted as the minimum operator acts as an 'or' gate.

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Ex: The metric d can be thought of as a binary predicate, since it describes how far away two objects in M are. If $d(x, y) = 0$, then it's "true" that they equal each other.

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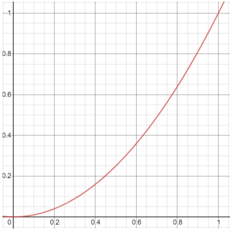
Ex: $f(x) = x^2$ is a function from $[0, 1] \to [0, 1]$.



Preliminaries on Continuous Logic

A **metric structure** $\mathfrak{M}$ on (M, d) is a tuple

$$\mathfrak{M} = (M, d, R_i, F_j, a_k : i \in I, j \in J, k \in K)$$

where the R's are predicates, the F's are functions, and the a's are designated points.

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Ex: $\mathfrak{M} = ([0,1], |\cdot|, P_{\mathbb{Q}} : [0,1] \to [0,1], \mathbb{Q} \cap [0,1])$ where $|\cdot|$ denotes the Euclidean metric and $P_{\mathbb{Q}} : [0,1] \to [0,1]$ rational polynomials whose range is bounded by $[0,1]$.

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For each distinguished point a there is a **constant symbol** t;

And a non-negative real number $D_{\mathcal{L}} = \sup\{d(a, b) : a, b \in M\}$, called the **diameter**.

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$`x_1', `\frac{1}{3}', `\left((x_4)^2 + \frac{1}{2}\right) \upharpoonright_{[0,1]}\ ' \in \mathrm{Term}_{\mathcal{L}}.$

Preliminaries on Continuous Logic

The set of **atomic formulas** of $\mathcal{L}$, denoted $\mathrm{Atom}_{\mathcal{L}}$, contains expressions of the form $P(t_1, ..., t_n)$ where P is a predicate symbol of $\mathcal{L}$ and $t_1, ..., t_n$ are terms. (Recall that d is a binary predicate symbol of $\mathcal{L}$.)

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Ex: '$|x_1 - 0|$', '$\left|\frac{2}{5} - \frac{1}{3}\right|$' $\in \mathrm{Atom}_{\mathcal{L}}$.

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(ii) if $u : [0,1]^n \to [0,1]$ is continuous and $\varphi_1, ..., \varphi_n$ are well-formed formulas, then $\mathbf{u}(\varphi_1, ..., \varphi_n)$ is a well-formed formula;

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(iii) if φ is a well-formed formula and x is a variable symbol, then '$\sup$'$_x\varphi$ and '$\inf$'$_x\varphi$ are well-formed formulas (normally we'll leave off the quotation marks).

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The set of all sentences of $\mathcal{L}$ is denoted by $S_{\mathcal{L}}$.

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Ex:

$$\left(\mathbf{d}(`\frac{1}{2}', `\frac{1}{4}')\right)^{\mathfrak{M}} = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4}$$

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For any $\varphi \in S_{\mathcal{L}}$, we say that $\mathfrak{M}$ **satisfies** φ, and write $\mathfrak{M} \models \varphi$, if $\varphi^{\mathfrak{M}} = 0$.

Preliminaries on Continuous Logic

At times it may be useful to consider the satisfiability of an arbitrary formula, rather than just sentences. Because of this, we may declare an **assignment** σ which assigns a unique point $a \in M$ to each variable $x \in V$. Satisfaction is defined naturally.

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(i) We call T a **theory** if T is a set of sentences of $\mathcal{L}$; i.e. $T \subseteq S_{\mathcal{L}}$.

(ii) We call a theory T **complete** if there is a $\mathcal{L}$-structure $\mathfrak{M}$ s.t. for every sentence φ, we have $\varphi \in T$ if and only if $\mathfrak{M} \models \varphi$.

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What the heck do we mean by computable from φ?

Effectivizing Continuous Logic

Recall that a **name** of a real number r is a sequence of natural numbers $\{k_n\}_{n=0}^{\infty}$ which serve as codes of a sequence of rational numbers $\{q_n\}_{n=0}^{\infty}$ s.t. $q_n \to r$ and for every n, for all $m > n$, $|q_n - q_m| < 2^{-(n+1)}$.

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A real number r is **computable** if there is some effective procedure which outputs a name for r; i.e., r is computable if there is a c.e. name for r.

Effectivizing Continuous Logic

Let $\mathcal{L}$ be a signature.

(i) Define **bounded subtraction** $\dot{-} : [0,1]^2 \to [0,1]$ as

$$x \dot{-} y := \begin{cases} x - y & \text{if } y \leqslant x \\ 0 & \text{otherwise.} \end{cases}$$

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(ii) Define **negation** $\neg : [0,1] \to [0,1]$ as

$$\neg x := 1 - x.$$

Effectivizing Continuous Logic

(iii) Define the **half operator** $\frac{1}{2} : [0, 1] \to [0, 1]$ as

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Note that all three of the above functions are continuous from $[0,1]^n \to [0,1]$. Hence for every $\varphi, \psi \in W_{\mathcal{L}}$, we have $\varphi \dot{-} \psi \in W_{\mathcal{L}}$, $\neg\varphi \in W_{\mathcal{L}}$, and $\frac{1}{2}\, \varphi \in W_{\mathcal{L}}$.

Effectivizing Continuous Logic

Theorem

(Corollary 1.6, Ben Yaacov and Usvyatsov, 2010) The set of functions $\{\dot{-}, \neg, \frac{1}{2}\}$ *generates, for every* $n \in \mathbb{N}$*, a dense subset* $Ł^n$ *of the set of all continuous functions* $C[0,1]^n$ *in the uniform convergence topology.*

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N.B. while $|W_{\mathcal{L}}| = 2^{\aleph_0}$, we have that $|Ł_{\mathcal{L}}| = \aleph_0$. This collapse to countability will be necessary for executing effective procedures on the set of formulas.

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Corollary

For every $\varphi \in W_{\mathcal{L}}$, there exists an Ł-name of φ.

Effectivizing Continuous Logic

Let $\varphi \in W_{\mathcal{L}}$.

(i) By a **name** of φ we mean a sequence of natural numbers $\{k_n\}_{n=0}^{\infty}$ which serve as codes of a sequence of Ł-neighborhoods $\{L_n\}_{n=0}^{\infty}$ which form an Ł-name of φ.

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(iii) Let T be a theory. We say that the map $\varphi \mapsto \varphi_T^{\circ}$ is **computable from** φ if there is some effective procedure which, given a name for φ, outputs a name for φ_T°.

Effectivizing Continuous Logic

A **computable presentation** of a metric structure $\mathfrak{M}$ is a pair $\mathfrak{M}^+ = (\mathfrak{M}, g)$ s.t. g maps $\mathbb{N}$ to some "dense" subset of M in a computable fashion, and all the functions and predicates of $\mathfrak{M}$ can be computed on the range of g.

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All the points in the range of g are called **rational points** of $\mathfrak{M}^+$, and computable notions relativize.

Classical Result

Theorem (Effective Completeness of (Classical) First-Order Logic)

(Millar, 1978, appears in Harizanov) Let $\mathcal{L}$ be a first-order language and and T be a decidable $\mathcal{L}$-theory. Then there is a decidable model $\mathfrak{M}$ s.t. $\mathfrak{M} \models T$.

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This (more-or-less) translates to the continuous case.

Previous Result

Theorem

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Now let us consider the same problem in terms of computable presentations.

Effective Completeness

Let $\Gamma \subseteq W_{\mathcal{L}}$ and $\mathcal{D}$ be the set of dyadic numbers, i.e. numbers of the form $\frac{k}{2^n} \leqslant 1$ for $k, n \in \mathbb{N}_0$.

Effective Completeness

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We say that Γ is **Henkin complete** if for every $\varphi \in Ł_{\mathcal{L}}$, every variable symbol x, and every pair $(p, q) \in \mathcal{D}$ s.t. $p < q$, there is some constant c s.t.

$$(\sup_x \varphi \dot{-} q) \wedge (p \dot{-} \varphi[c/x]) \in \Gamma.$$

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If we can extend T to a maximally consistent and Henkin complete set, terms built up from these constants can form the rational points of a computable presentation of a metric space $\mathfrak{M}$ which models T.

Effective Completeness

Lemma

Any complete, decidable theory T can be extended to a maximally consistent and Henkin complete set Λ where there is a uniformly computable sequence $\{\Lambda_s\}_{s\in\omega}$ of finite sets s.t. $\Lambda = \bigcup_{S\in\omega}\bigcap_{S\leqslant s<\omega}\Lambda_s$.

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Lemma

Any complete, decidable theory T can be extended to a maximally consistent and Henkin complete set Λ where there is a uniformly computable sequence $\{\Lambda_s\}_{s\in\omega}$ of finite sets s.t. $\Lambda = \bigcup_{S\in\omega}\bigcap_{S\leqslant s<\omega}\Lambda_s$.

Proof sketch: Throw in a bunch of constants at every stage so all the Henkin business works nicely.

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Proof sketch: Throw in a bunch of constants at every stage so all the Henkin business works nicely. At each stage s check all wff's φ of length less than s for relative consistency up to an error term of 2^{-s}. For each of these, put them into Λ_s. These will all be effective (actually each is finite), but in the limit converge to a maximally consistent, Henkin complete extension.

Effective Completeness

Theorem (Effective Completeness of Continuous First-Order Logic)

Let $\mathcal{L}$ be a signature and T be a complete, decidable $\mathcal{L}$-theory. Then there is a computably presentable metric structure $\mathfrak{M}$ s.t. $\mathfrak{M} \models T$.

Effective Completeness

Proof sketch. Extend T to a Henkin complete and maximal consistent set Λ as outlined in the previous lemma.

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Let $\mathfrak{M}^*$ be the model of T whose universe M^* is the set of closed terms in C^* (the Henkin constants), and interpret functions, predicates, and terms as themselves.

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Let $\mathfrak{M}^*$ be the model of T whose universe M^* is the set of closed terms in C^* (the Henkin constants), and interpret functions, predicates, and terms as themselves.

Then $\mathfrak{M}^* \models T$. Further we can quotient out by the equivalence relation $t_0 \sim t_1$ iff $d(t_0, t_1) = 0$ to get a metric structure $\mathfrak{M}$ s.t. $\mathfrak{M} \models T$.

Effective Completeness

Now let g be any map which witnesses the computability of M^*, and define the presentation $\mathfrak{M}^+ := (\mathfrak{M}, g)$.

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This presentation is computable since the rational points of M^+ are precisely $(M^*/\sim)$ and we can compute the value of $P^{\mathfrak{M}^*}(t_1, ..., t_{n(P)})$ up to arbitrary precision, by checking a large enough Λ_n from the previous proposition. '■'

Computational vs. Syntactic Complexity

Let $\mathfrak{M}$ be a computably presentable $\mathcal{L}$-structure computable presentation $\mathfrak{M}^\sharp$. We define the **atomic diagram** of $\mathfrak{M}$, denoted $\mathfrak{D}_{\mathfrak{M}}$, to be the set of atomic Ł-sentences or negated atomic Ł-sentences with only constant symbols interpreted as computable points of $\mathfrak{M}^\sharp$, which are satisfied by $\mathfrak{M}$.

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Proposition

The atomic diagram of a computably presentable model is Π^0_1.

Computational vs. Syntactic Complexity

Proof. Let $\{\theta_k\}_{k=1}^{\infty}$ be an effective enumeration of the atomic and negated atomic Ł-sentences of $\mathcal{L}$ which only include the constant symbols interpreted as computable points of $\mathfrak{M}^\sharp$. Notice

$$\theta_s \in \mathfrak{D}_{\mathfrak{M}} \iff \forall n \in \mathbb{N},\ \mathfrak{M} \models \theta_s \dot{-} 2^{-n}. \blacksquare$$

Classical "Result"

Definition

(Harizanov, 1998) A (classical) model $\mathfrak{M}$ is computable if M is computable and some form of its atomic diagram is decidable.

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Does this translate to the continuous case? Not necessarily! Computably presentable structures can have non-computable atomic diagrams!

Computational vs. Syntactic Complexity

Lemma

There exists a uniformly computable sequence of real numbers $\{c_n\}_{n=0}^{\infty}$ *s.t.* $c_n \leqslant \frac{1}{2}$ *if and only if* $n \in K^c$.

Computational vs. Syntactic Complexity

Theorem

There exists a computably presentable model $\mathfrak{M}$ s.t. its atomic diagram $\mathfrak{D}_{\mathfrak{M}}$ is Π^0_1-complete.

Computational vs. Syntactic Complexity

Proof sketch. Let $\mathfrak{M} = ([0,1], |\cdot|, P_{\mathbb{Q}} : [0,1] \to [0,1], \mathbb{Q} \cap [0,1])$, where $P_{\mathbb{Q}} : [0,1] \to [0,1]$ denotes rational polynomials with domain $[0,1]$ and range bounded by $[0,1]$. Notice that this is a metric structure on the bounded metric space $([0,1], |\cdot|)$.

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Now fix $\{c_n\}_{n=0}^{\infty}$ a uniformly computable sequence of real numbers s.t. $c_n \leqslant \frac{1}{2}$ if and only if $n \in K^c$, which exists by the previous lemma, and throw them all in as as rational points for a computable presentation $\mathfrak{M}^\sharp$ of $\mathfrak{M}$.

Computational vs. Syntactic Complexity

We can now check the atomic diagram of $\mathfrak{M}$ for

$$\begin{aligned} c_n \leqslant \frac{1}{2} &\iff |c_n| \dot{-} \frac{1}{2} = 0 \\ &\iff \left|c_n \dot{-} \frac{1}{2}\right| = 0 \\ &\iff \big(d(0, f(\mathbf{c}_n))\big)^{\mathfrak{M}} = 0 \\ &\iff \mathfrak{M} \models d(0, f(\mathbf{c}_n)) \end{aligned}$$

where f is the function symbol corresponding to the bounded rational polynomial $x \dot{-} \frac{1}{2}$.

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where f is the function symbol corresponding to the bounded rational polynomial $x \dot{-} \frac{1}{2}$. This allows us to decode K^c. '■'

Computational vs. Syntactic Complexity

Let $\mathcal{L}$ be a signature.

(i) We define the sets $\Sigma_0^0(\mathcal{L}) = \Pi_0^0(\mathcal{L}) = \Delta_0^0(\mathcal{L})$ as the set of all quantifier-free Ł-formulas of $\mathcal{L}$;

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(iii) for every $n \in \mathbb{N}$, we define $\Pi^0_n(\mathcal{L})$ to be the set of all Ł-formulas of the form $\sup_{\overline{x}} \varphi$ s.t. $\overline{x}$ is a tuple of variable symbols and $\varphi \in \Sigma^0_{n-1}$.

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$\Sigma_n^0(S_{\mathcal{L}})$, $\Pi_n^0(S_{\mathcal{L}})$, and $\Delta_n^0(S_{\mathcal{L}})$ relativize as sets of Ł-sentences of $\mathcal{L}$ in the standard way.

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Theorem

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Computational vs. Syntactic Complexity

For each $n \in \mathbb{N}$,

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At stage s, recall that since $\mathfrak{M}^\sharp$ is computable and φ_s is quantifier-free containing only constant symbols interpreted as computable points of $\mathfrak{M}^\sharp$, we can compute a Cauchy name for $\varphi_s^{\mathfrak{M}}$; call it $\{q_m\}_{m=1}^{\infty}$.

Computational vs. Syntactic Complexity

Notice that

$$\begin{aligned} \varphi_s^{\mathfrak{M}} \leqslant r_s &\iff \forall m \in \mathbb{N},\ \varphi_s^{\mathfrak{M}} < r_s + 2^{-m} \\ &\iff \forall m \in \mathbb{N},\ q_m < r_s + 2^{-m}, \end{aligned}$$

with this condition being Π_1^0.

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$$\varphi_s^{\mathfrak{M}} < r_s \iff \exists n \in \mathbb{N},\ \varphi_s^{\mathfrak{M}} < r_s - 2^{-n}$$
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(ii) and (iii) relativize similarly. '■'

Moving Forward

(i) Generalize optimal bound results to further diagrams (upper bounds already proven).

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(ii) Extend complexity results to infinitary formulas.

Bibliography

Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, "Model theory for metric structures," *Model Theory with Applications to Algebra and Analysis*, vol. 2, (Zo´e Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, editors) London Math Society Lecture Note Series, vol. 350, London, 2008, pp. 315-427.

Itai Ben Yaacov and Arthur Paul Pedersen, "A proof of completeness for continuous first-order logic," *The Journal of Symbolic Logic*, vol. 75 (2010), no. 1, pp. 168-190.

Itai Ben Yaacov and Alexander Usvyatsov, "Continuous First Order Logic and Local Stability," *Transaction of the American Mathematical Society*, vol. 362, no. 10, 2010, pp. 5213-5259.

Bibliography

Klaus Weihrauch, *Computable Analysis: An Introduction*, Texts in Theoretical Computer Science. An EATCS Series, Springer, 2000.

V. S. Harizanov, "Pure computable model theory," *Handbook of Recursive Mathematics*, vol. 1, (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J.B. Remmel, and V. W. Marek, editors) Studies in Logic and the Foundations of Mathematics, 138-139, North-Holland, Amsterdam, 1998, pp. 3-114.

Wesley Calvert, "Metric structures and probabilistic computation," *Theoretical Computer Science*, vol. 412 (2011), pp. 2766-2775.

Questions?

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Or comment on this video!