Henkin models in the continuum
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The Cantor and Shelah Elevators

In the novel *White Light* [**?**], Rudy Rucker proposes a metaphor for the continuum hypothesis. One can reach $\aleph_1$ by a laborious climb up the side of Mt. ON, pausing at ϵ_0.

Or one can take Cantor's instantaneous elevator through the center of the mountain.

Here, working in ZFC, we take Shelah's elevator, which is a bit slower. After countably many floors, each with finitely many rooms, we reach an object of cardinality $2^{\aleph_0}$.

The underlying construction applies for finding atomic models, two-cardinal theorems, a collection of continuum many points that are *asymptotically similar* (a weak form of indiscernibility), and a coloring with a Borel square of size continuum.

Acknowledgements

Joint work with Chris Laskowski, built on Shelah.

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Constructing many atomic models in $\aleph_1$.
Journal of Symbolic Logic, 81:1142–1162, 2016.

Henkin's Proof

Completeness Proof: Henkin

Definition

T has the *witness property* if for every formula $\phi(x)$, there is a *witness constant* c_ϕ (in an expanded language) such that

$$T \vdash (\exists x)\phi(x) \rightarrow \phi(c_\phi).$$

Begin with a consistent first order T. Inductively

Step 0

Extend T to one satisfying the witnessing property.

Step 1

Add ϕ_α or $\neg\phi_\alpha$ at stage α to ensure that each sentence is decided.

Simpler: We move from a countable model M of T to a model with $2^{\aleph_0}$ elements AND omitting certain (all) non-principal types.

Atomic Models and $L_{\omega_1,\omega}$

Atomic models

Let T be a complete first order theory in a countable language.

Definition

A model M of T is atomic if every finite tuple from M realizes a principal type.

Examples

1. $(\mathbb{Q}, <)$, $(\Re, <)$
2. $(\mathbb{Z}, <)$
3. $(N, +, \times)$

Not two copies of $(\mathbb{Z}, <)$ if $\tau = \langle < \rangle$.

Questions

Question

Does the Löwenheim-Skolem theorem hold for ‘atomic’ models? down? up?

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Does the Löwenheim-Skolem theorem hold for 'atomic' models? down? up?

Answer

Not up, but, if T has an atomic model of cardinality $\beth_{\omega_1}$ then it has arbitrarily large atomic models.

Note: free if the first order T is $\aleph_0$-categorical.

complete $L_{\omega_1,\omega} \leftrightarrow$ atomic first order

Theorem:Chang/Lopez-Escobar

For every τ-sentence ψ $L_{\omega_1,\omega}$ in a countable vocabulary τ.
there is a countable vocabulary τ' extending τ, and a theory T such that:
$\mathrm{mod}(\psi)$ is the reducts of $\mathrm{mod}(T)$.

complete $L_{\omega_1,\omega}$-sentences

Definition

ϕ is $L_{\omega_1,\omega}$-complete if for every $\psi \in L_{\omega_1,\omega}$, $\phi \models \psi$ or $\phi \models \neg\psi$.

A τ-structure M is $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}$-types (over the empty set).

Generalized Scott's theorem

A structure satisfies a complete sentence of $L_{\omega_1,\omega}$ if and only if it is small.

Reducing complete to atomic

The models of a complete sentence ϕ in $L_{\omega_1,\omega}$ can be represented as: $\mathrm{mod}(\phi)$ is the class of atomic models (realize only principal types) of a complete first order theory (in an expanded language).

Hjorth's theorem

Theorem: Hjorth

There is a sequence of countable families Φ_α, $\alpha < \omega_1$, of complete sentences of $L_{\omega_1,\omega}$, such that some $\phi \in \Phi_\alpha$ has a model in $\aleph_\alpha$ and no bigger.

Theorem: B-Koerwien-Laskowski

There is a sequence ϕ_n of complete sentences of $L_{\omega_1,\omega}$, $n < \omega$, such that ϕ_n has a model in $\aleph_n$ and no bigger.

Questions

Can the second theorem be extended to all countable α?

What about finding atomic models in the continuum?

Henkin models in the continuum: Notation

Let L be any countable language. Let Z be an indexed, distinguished set of variable symbols.

For ϕ any L-formula with at most k free variables and for any set of variables Z, we introduce the notion of Z-instantiated formula.

For any $(z_1, \ldots, z_k) \in Z^k$, let $\phi(z_1, \ldots, z_k)$ be the result of substituting the variable symbol z_j for the jth free variable for each j.

We call $\phi(z_1, \ldots, z_k)$ a *Z-instantiated formula*.

$Fm(Z)$ denotes the set of all formulas obtained by this procedure. We will use several choices for Z.

A witnessed Henkin set

Definition

A witnessed Henkin set is a subset $\mathcal{H} \subseteq Fm(Z)$ such that:

- **Satisfiable:** If $\phi(z_1, \ldots, z_k) \in \mathcal{H}$, then there is some L-structure N and $(a_1, \ldots, a_k) \in N^k$ such that $N \models \phi(a_1, \ldots, a_k)$.
- **Completeness:** For every $\phi \in Fm(Z)$, exactly one of $\phi, \neg\phi \in \mathcal{H}$; and
- **Henkin witnesses:** If $\exists w \phi \in Fm(Z)$, then either $\neg\exists w \phi(w) \in \mathcal{H}$ or $\phi(z^*) \in \mathcal{H}$ for some $z^* \in Z$.

A witnessed Henkin set gives a model

If $\mathcal{H} \subseteq Fm(Z)$ is a witnessed Henkin set, then there is a unique L-structure M with universe $Z/\sim$ that satisfies

$$M \models \phi([z_1], \ldots, [z_k]) \iff \phi(z_1, \ldots, z_k) \in \mathcal{H}.$$

In particular, there is an $L - \{=\}$ structure M' universe A and $\sim$ is a congruence on M.

The entire Henkin set $\mathcal{H}(Z)$ determines a canonical L-structure M with universe $Z/\sim$.

Specifying the $M(s)$

Let Z be an indexed, distinguished set of variable symbols. $Z = X \cup Y$, where,
X is indexed as $\{x_\eta : \eta \in 2^\omega\}$ and
Y as $\{y_{s,i} : s\}$ for s a finite subset of 2^ω, $i \in \omega$.

$M(s)$ has domain $\{x_\eta : \eta \in s\} \cup \{y_{t,i} : t \subset s, i < \omega\}$.

The y variables close $M(s)$ to be a model.
For example, if we are constructing an abelian group, and $s = \{\eta, \eta'\}$, then for some $i \in \omega$, $\mathcal{H}$ would include the instantiated formula $x_\eta + x_{\eta'} = y_{\{\eta,\eta'\},i}$.

Trivial dcl

In this case can omit the y's but need more x's, $x_{\eta,i}$ for $i < \omega$.

The uncountable model

The entire Henkin set $\mathcal{H}(Z)$ determines a canonical L-structure M with universe $Z/\sim$.
Since any finite tuple $\mathbf{z}$ from Z is contained in some Z_s, M can be identified with

$$M = \bigcup\{M(s) : s \text{ a finite subset of } 2^\omega\}$$

- $M(s) \preceq M$ for each finite $s \subseteq 2^\omega$, hence $M \models T$ if and only if some (equivalently, every) $M(s) \models T$;
- For Δ any partial type, M omits Δ if and only if every $M(s)$ omits Δ; so
- M is atomic if and only if every $M(s)$ is atomic.

Trivial DCL

dcl-triviality

Definition

For an L-structure M and $A \subseteq M$, $b \in M$ is *A-definable* if there is a formula $\phi(x, \boldsymbol{a})$ with $\boldsymbol{a}$ from A with unique solution b in M.

The *definable closure*, $\mathrm{dcl}(A)$, is the set of A-definable elements of M.

For *A-algebraic* and $\mathrm{acl}(A)$ replace unique by finite.

Clearly, $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ for any subset $A \subseteq M$. We distinguish structures for which both of these closures are trivial.

Definition

Fix a countable language L. An L-structure M has trivial definable closure *(is* dcl*-trivial) if* $\mathrm{dcl}(A) = \mathrm{acl}(A) = A$ *for* **every** *subset* $A \subseteq M$.

Note L can have no function or constant symbols.

Splitting the index set

Notation for dealing with sequences from 2^ω

Definition

Fix an integer ℓ.

- *A k-tuple $(\eta_0, \ldots, \eta_{k-1})$ of distinct elements from 2^ω* splits by ℓ *if the restrictions $\{\eta_i\restriction_\ell : i < k\}$ to 2^ℓ are distinct.*
- *Two k-tuples $(\eta_0 \ldots, \eta_{k-1})$ and $(\tau_0, \ldots, \tau_{k-1})$ of distinct elements from 2^ω are* similar (mod ℓ) *if $(\eta_0, \ldots, \eta_{k-1})$ splits by ℓ and $\eta_i\restriction_\ell = \tau_i\restriction_\ell$ for each $i < k$.*

Clearly, every k-tuple of distinct elements from 2^ω splits by some ℓ, and consequently splits by every $\ell' \geq \ell$; and similarity (mod ℓ) is an equivalence relation on the set of k-tuples from 2^ω that split by ℓ.

Asymptotic Similarity

Definition

Fix an L-structure M. A subset of M, indexed by $\{a_\eta : \eta \in 2^\omega\}$*, is* asymptotically similar *if, for every k-ary L-formula* θ*, there is an integer* N_θ *such that for every* $\ell \geq N_\theta$*,*

$$M \models \theta(a_{\eta_0}, \ldots, a_{\eta_{k-1}}) \leftrightarrow \theta(a_{\tau_0}, \ldots, a_{\tau_{k-1}})$$

whenever $(\eta_0, \ldots, \eta_{k-1})$ *and* $(\tau_0, \ldots, \tau_{k-1})$ *are similar (mod* ℓ*).*

Example: formula by formula indiscernibility

The structure $M = (2^\omega, U_a)_{a \in 2^{<\omega}}$, where each U_a is a unary predicate interpreted as the cone above a, i.e.,

$$U_a(M) = \{\eta \in 2^\omega : a \lhd \eta\}.$$

The entire universe of M, $\{\eta : \eta \in 2^\omega\}$, is asymptotically similar.

Building dcl-trivial structures in $2^{\aleph_0}$

Theorem

Suppose M is a dcl-trivial structure in a countable language L. There is a Borel model N (atomic if M is) elementarily equivalent to M of size continuum that satisfies:

1. The universe of N is indexed as $2^\omega \times \omega$;
2. The universe of N can be partitioned as $N = \bigcup_{i\in\omega} A_i$, where, for each i, $A_i = \{a_{\eta,i} : \eta \in 2^\omega\}$ is an asymptotically similar subset;
3. If we place the usual measure on $2^\omega \times \omega$, then for every k, every non-degenerate definable subset of N^k has positive measure (with respect to the product measure on $(2^\omega \times \omega)^k$. But points have measure 0.
4. Also can omit meager sets of non-principal types.

Variables and Commitments

An *fmac* A is a finite maximal antichain in $2^{<\omega}$, e.g. 2^n

Two sets of variables

1. For each $t \in 2^{<\omega}$ and $i < \omega$ there is a variable $x_{t,i}$; $x_{t,i} \in X_A$ if t is in the fmac A.
2. For each $\eta \in 2^{\omega}$ and $i < \omega$ there is a variable $x_{\eta,i}$; X is the set of such variables.

Commitments

1. For an fmac A, an A-commitment is a formula $\phi(\mathbf{x})$ where $\mathbf{x} = \langle x_{a,i} : a \in A, i < k_\phi \rangle$.
2. A *lifting* $h^* : A \to 2^\omega$ of the fmac A to 2^ω is a mapping satisfying $a \triangleleft h^*(a)$ for every $a \in A$.
   Note that any lifting h^* naturally induces an injection, also dubbed h^*, where $h^*(s) = \{h(a) : a \in s\}$

$$h^* : Fm(X_A) \to Fm(X).$$

Purpose of commitments

Notation

1. $\mathbb{P}_A$ is the set of A commitments.
2. $A, B \subseteq 2^{<\omega}$. B covers A if each $a \in A$ extends to at least one element of B.
3. $(A, \phi) \leq (B, \psi)$ iff B covers A and $\psi \to h(\phi)$ for each lifting $h : A \to B$.

The crux is that in our construction when we make an A-commitment (A, ϕ) we are guaranteeing to include in a witnessed Henkin set $\mathcal{H}$, each formula $h(\phi)$ for each lifting h of A.

Splitting the formulas

Definition

Given any fmac A and any $a \in A$, the *splitting of A at a* is the fmac $A^{*a} = A \setminus \{a\} \cup \{\hat{a}0, \hat{a}1\}$.
There are two liftings $h_0, h_1 : A \to A^{*a}$, distinguished by $h_i(a) = \hat{a}i$ for $i = 0, 1$.

Splitting For every $a \in A$ there is an A^{*a}-commitment $\phi^* \in \mathbb{P}$ extending ϕ. [In particular, $\phi^* \vdash h_0(\phi) \wedge h_1(\phi) \wedge x_{\hat{a}0} \neq x_{\hat{a}1}$.]

Lemma

If there is system of commitments $\mathbb{P}$ that satisfies splitting and produces an atomic witnessed Henkin set, then there is an atomic model in the continuum.

dcl-Proof Sketch 1

The *key property* of a dcl-trivial structure M:
If $M \models \exists u \phi(u, \mathbf{c}) \wedge u \notin \mathbf{c}$, then $\phi(u, \mathbf{c})$ has infinitely many solutions in M.

Henkin witnesses

Easy if $\exists u \theta(u, \mathbf{c})$ is not satisfied or satisfied in $\mathbf{c}$.
Suppose $M \models \exists u \theta(u, \mathbf{c}) \wedge \bigwedge u \notin \mathbf{c}$. Then, by the key property of dcl-triviality, choose $b^* \in M \setminus \mathbf{c}$ such that $M \models \theta(b^*, \mathbf{c})$. Choose any $a \in t(\mathbf{z})$ and $j \in \omega$ such that $x_{a,j} \notin \mathbf{x}$ and put

$$\phi(x_{a,j}\mathbf{x}) := \phi(\mathbf{x}) \wedge \psi(x_{a,j}\mathbf{z}) \wedge \bigwedge x_{a,j} \notin \mathbf{x}$$

Then $b^*\mathbf{c}$ witnesses that $\phi^* \in \mathbb{P}_A$, which visibly extends ϕ.

dcl-Proof Sketch 2

Splitting

Choose any $a \in A$. To handle this case, we start with a Claim, whose proof is an easy induction on k; the key property yields the case $k = 1$:

Claim. For every $k \geq 1$, for every $\phi(\mathbf{x}) \in \mathcal{D}_0$, and for every partitioning of $\mathbf{x} = \mathbf{uv}$ with $\lg(\mathbf{u}) = k$, then for every $\mathbf{b}$ from M such that $M \models \exists \mathbf{u} \phi(\mathbf{u}, \mathbf{b})$, there is an infinite, pairwise disjoint set $\{\mathbf{c}_j : j \in \omega\} \subseteq M^k$ of realizations of $\phi(\mathbf{u}, \mathbf{b})$.

Assume Claim

Partition the variables of $\phi(\mathbf{x})$ into two disjoint subsequences $\mathbf{x} = \mathbf{x}_a \mathbf{x}^*$, where $\mathbf{x}_a$ consists of all $x_{a,i} \in \mathbf{x}$, while $\mathbf{x}^*$ consists of all $x_{a',i} \in \mathbf{x}$ with $a' \neq a$. This partition induces a partition of our realizing sequence $\mathbf{c}$ into $\mathbf{c}_a \mathbf{b}$, where $\mathbf{c}_a$ corresponds to $\mathbf{x}_a$, while $\mathbf{b}$ corresponds to $\mathbf{x}^*$. Put

$$\phi^*(\mathbf{x}_{a\frown 0}, \mathbf{x}_{a\frown 1}, \mathbf{x}^*) := \phi(\mathbf{x}_{a\frown 0}, \mathbf{x}^*) \wedge \phi(\mathbf{x}_{a\frown 1}, \mathbf{x}^*) \wedge \text{ `}\mathbf{x}_{a\frown 0}, \mathbf{x}_{a\frown 1}, \mathbf{x}^* \text{ are distinct'}$$

Then the Claim implies that $(A^{*a}, \phi^*) \in \mathbb{P}_{A^{*a}}$, and is as required.

There is a measure concentrating on (copies of) any M with Trivial DCL

Borel Structure to Logic topology

Notation

1. For a countable relational language L, $\mathrm{mod}_\omega L$ denotes the Polish space of countable L-structures.
2. Let N be a Borel L-structure, N has trivial dcl.
   Define $\mathcal{F}_N : N^\omega \to \mathrm{mod}_\omega L$ by $\mathcal{F}_N(\boldsymbol{a})$ is the induced structure on $\boldsymbol{a}$.
   For a continuous (points have measure zero) probability measure m on $\Re$, let
   $$\mu_{N,m} = m^\infty \circ \mathcal{F}_N^{-1}.$$
   The key property of trivial dcl guarantees continuity.

Lemma (AFP 3.6)

Let N be a relational Borel L-structure and let m be a continuous probability measure on N. Then $\mu_{N,m}$ is a probability measure on $\mathrm{mod}_\omega L$ that concentrates on the isomorphism classes of countable infinite substructures of N.

An alternate proof of the AFP theorem

AFP: Ackerman, Freer, Patel

Theorem (AFP)

If ϕ is a complete sentence in $L_{\omega_1,\omega}$ whose countable model M has trivial dcl, then there is a continuous probability measure on $\mathrm{mod}_\omega(L)$ that concentrates on the orbit of M.

Proof.

1. Find a Borel model of ϕ in the continuum and a continuous measure.
2. Apply last slide.

We have given a direct proof of item 1.

Still another proof of the AFP theorem

Added in proof: I learned the following proof during a rehearsal for this talk.

Andrew Marks
`https://www.math.ucla.edu/~marks/notes/afp_v2.pdf`
gave a direct proof using the Baire Category theorem the AFP theorem.

This shortcuts the trip through the model of the continuum and so misses transfer step 2 which has independent interest of the AFK proof.

The use of the Baire category theorem seems to be the essential combinatorics of our argument.

General Case

The general case

Theorem

Let T be any theory in a countable language. If there is a sufficiently dense, partially ordered set $(\mathbb{P}, \leq)$ of commitments that are each satisfied in a model of T, then there is a Borel model M of T of size continuum with an asymptotically similar subset $\{a_\eta : \eta \in 2^\omega\}$.
Moreover:

1. If T satisfies the omitting types condition, the M omits the relevant types.
2. If T is complete with an atomic model then M is an atomic model of T.

Finding witnesses

New variables for the diagram

1. For each $t \in 2^{<\omega}$ and $i < \omega$ there is a variable x_t. $x_t \in X_A$ if t is in the fmac A.
   Add variables $Y_A = \{y_{S,i} : S \subseteq A, i < \omega\}$.
2. For each $\eta \in 2^{\omega}$ and $i < \omega$ there is a variable $x_{\eta,i}$. X is the set of such variables.
   Add variables $Y = \{y_{S,i} : S \subseteq_{\omega} 2^{\omega}, i < \omega\}$.

The purpose of the $y_{S,i}$ is to witness a formula $\exists y \phi(\mathbf{x}, y)$ where the indices of variables in $\mathbf{x}$ come from S.

Formula-based closure relations

Definition

Let M be any L-structure. A formula-based closure relation on M *is a closure relation on M such that whenever $a \in \mathrm{cl}(B)$, t there is a finite tuple $\mathbf{b}$ from B and a formula $\phi(x, \mathbf{y}) \in \mathrm{tp}(a\mathbf{b})$ such that $a' \in \mathrm{cl}(\mathbf{b}')$ whenever $M \models \phi(a', \mathbf{b}')$.*

Examples include *equality* $(M, =)$, where $\mathrm{cl}(A) = A$ for all $A \subseteq M$,

definable closure (M, dcl),

algebraic closure (M, acl)

and NEW:
pseudo-algebraic closure (M, pcl), which is well behaved whenever M is atomic.

Sufficient pregeometries

Definition

A formula-based closure relation (M, cl) *on is called* sufficient *if the following additional conditions hold:*

1. *'Exchange:' i.e., if* $a \in \mathrm{cl}(Bc) \setminus \mathrm{cl}(B)$*, then* $c \in \mathrm{cl}(Ba)$*;*
2. *'Extendible There is* $a \in M \setminus \mathrm{cl}(\emptyset)$*; and*
3. *'Weak homogeneity:' For all finite* **b** *and L-formulas* $\phi(w, \mathbf{b})$*,*
   if there is $a \notin \mathrm{cl}(\mathbf{b})$ *with* $M \models \phi(a, \mathbf{b})$*,*
   then for every finite $E \subseteq M$*, there is* $a' \notin \mathrm{cl}(E)$ *that also satisfies* $M \models \phi(a', \mathbf{b})$*.*

Main Theorem

Theorem

Suppose (M, cl) is a sufficient pregeometry.
Then there is a Borel model $N \equiv M$ of size continuum with a cl-independent, asymptotically similar subset $\{a_\eta : \eta \in 2^\omega\}$ from N.

Moreover, if M is atomic (with respect to $Th(M)$) then we may additionally choose N to be atomic.

More generally, if $\{\Delta_m(\mathbf{w}_m) : m \in \omega\}$ is a countable set of partial types, each of which is omitted in M, then we may additionally require that N omits every Δ_m.

Pseudo-closure

Definition

1. *Let M be an atomic model and suppose $a, \mathbf{b}$ are from M. We say a is* pseudo-algebraic over $\mathbf{b}$ in *M, written $a \in \mathrm{pcl}(\mathbf{b})$, if every elementary substructure $N \preceq M$ that contains $\mathbf{b}$ also contains a.*
2. *Let M be an atomic model and suppose T has an uncountable atomic model. We say that T is* pseudominimal *if (M, pcl) satisfies Exchange for some/every atomic model M of T.*

   That is, for every finite set C from M and elements $a, b \in M$, if $b \in \mathrm{pcl}(Ca)$ but $b \notin \mathrm{pcl}(C)$, then $a \in \mathrm{pcl}(Cb)$.

Theorem

if T has an uncountable atomic model that is pseudo-minimal, then there is an atomic model of T in the continuum.

Application

Theorem: B-Laskowski-Shelah

If T has has $< 2^{\aleph_1}$ atomic models of cardinality $\aleph_1$ then the pseudo-minimal types are dense.

Question: B-Laskowski-Shelah

If T has has $< 2^{\aleph_1}$ atomic models of cardinality $\aleph_1$, must T have an atomic model of T in the continuum?